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THE ONE-ON-ONE STOCHASTIC DUEL: PARTS I AND II

INTERIM TECHNICAL REPORT

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This report is approximately one half the output of a project to exhaustively review the literature of one-on-one stochastic duels. It contains two main parts. The first is an exposition, identifying and tying together for the first time, by examples, the two principal analytical techniques used, namely; mixture theory and renewal theory. Secondly, it contains an exhaustive, comprehensive, and fully annotated bibliography of all research results in this theory which are known to the author. Finally, is also lists many theorems involving characteristic functions which are particularly useful →

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FOREWORD

The stochastic duel was first formulated and analyzed by G. Trevor Williams and the author, in a paper presented on 11 May 1966 at the 4-th Annual ORSA meeting in Washington, D.C. Since then, about fifty research papers have been written by authors scattered in many parts of the world. This fairly substantial body of results has proven of value to many military OR analysts, both in and out of the services. However, much of it is inaccessible, or mathematically obtuse. Consequently, we have set out to remedy this situation, with the support of the U. S. Army Research Office.

This report is approximately one-half of the total output of the project. The material prepared so far is sufficiently self-contained as to warrant issuance at this time. The remainder of the report will be forthcoming within the next year, at which time, the entire manuscript will be updated and brought together as one entity.

The present work consists, primarily, of two parts. In Part I an exposition of the two principal methods of deriving results is given. These are the mixture technique and the semi-Markov terminating renewal process technique. For the first time, to the best of the author's knowledge, it is carefully and explicitly shown that, in fact, these are the techniques being used and precisely how they are being used; and, finally, how they are related. We hope to clear up much of the obscurity and mystery surrounding the utility of the techniques and also to indicate when there is (or is not) an

advantage to using one or the other. This is done, primarily, by carrying an example through each technique.

Part II is a comprehensive, exhaustive and fully annotated bibliography of all research papers on the topic, known to the author. It has been formatted to uniformly display, in detail, what has been accomplished in each paper.

The additional work, to be done later, will include:

(1) an historical and expository introduction;

(2) possibly, some further exposition on techniques, and, most importantly,

(3) a comprehensive compendium of all known results reduced to a common notation and organized to facilitate the location of any desired result.

It is believed that this project will

(1) make all results easily and conveniently available to the analyst, and

(2) will aid the research worker in identifying what has been done (to prevent reinventing the wheel) and what has not been done, so that he may direct his efforts in a productive direction.

The author would be grateful to learn of any inadvertant mistakes, omissions, or other errors which may have occurred.

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PART I

AN EXPOSITION ON TECHNIQUES

1. INTRODUCTION

In the fundamental duel (FD), two contestants, A and B, fire at each other at certain intervals and either hit or miss on each round fired. The duel terminates when either, or both, are hit. The hit probabilities for each are constant from round to round and, in general, are different for each. The time between rounds may be a continuous random variable or may be a constant, and, in general, is different for each. The contestants both start at time zero with unloaded weapons and fire their first rounds sometime later. They both have unlimited ammunition and unlimited time.

The analysis begins by first considering a marksman firing at a passive target, under the same conditions given above, until he hits it. This is called the fundamental marksman problem (FM). From this we may solve the duel problem by considering each marksman to be firing independently of the other. The first to hit his passive target wins. This is entirely equivalent to the fundamental duel, as the model described above in no way links the actions of one duelist to those of the other.

Consequently, let us first consider the case of the marksman versus a passive target, and further, let us confine ourselves here to the situation where his interfiring time is a continuous random variable. It is clear that he might hit on his first round fired, or possibly on the second, or, in fact, on any round, providing he has failed to hit on all preceding rounds. That is, if his hit probability is p, $(q = 1 - p)$, then the probability that his first (and fatal) hit is on the n^{th} round and is $p\,q^{n-1}$. If his interfiring time has a probability density function (pdf)

given by $f(t)$, then $f(t)$ is also his pdf to a hit on the first round. If he hits on the second round after failing on the first, his pdf to a hit is the convolution of $f(t)$ with itself, denoted by $f*f(t)$. This is because if he has fired twice, he has made two selections (at random) from $f(t)$ and added them together to determine his time to fire the second round. Continuing, his pdf of time to fire the n^{th} round is given by $f*f*f* \ldots *f(t) = f^{n*}(t)$, i.e., n convolutions of $f(t)$ with itself.

The stochastic process described above is called a mixture and may be described as follows. Let T be the continuous random variable, time to a hit, and let X_i, $i = 1,2,3,\ldots$, be the continuous random variables, time between firing epochs, where the X_i are known, independent and identically distributed (iid) as X with pdf $f(t)$. Then

$$\left.\begin{array}{lll} T = X_1 & \text{with probability} & p \\ \quad = X_1 + X_2 & \quad " & pq \\ \quad \vdots & \quad \vdots & \vdots \\ \quad = X_1 + X_2 + X_3 + \cdots + X_n & \quad " & pq^{n-1} \\ \quad \vdots & \quad \vdots & \vdots \end{array}\right\} , \quad (1)$$

where all the rows above are the mutually exclusive and exhaustive ways of obtaining a hit. Now, let $h(t)$ be the pdf of T and $f(t)$ be the pdf of each X_i, then by the basic property of mixtures:

$$P[t < T < t + dt]$$

$$= h(t)dt = pf(t)dt + pqf^{2*}(t)dt$$

$$+ pq^2 f^{3*}(t)dt + \cdots + pq^{n-1} f^{n*}(t)dt + \cdots$$

or

$$h(t) = \sum_{n=1}^{\infty} pq^{n-1} f^{n*}(t) \quad . \tag{2}$$

This expression can be greatly simplified by converting each side to a characteristic function (Fourier transform) as follows:

$$\Phi(u) = \int_0^{\infty} e^{iut} h(t)dt \quad \text{and} \quad \varphi(u) = \int_0^{\infty} e^{iut} f(t)dt \quad . \tag{3}$$

Using the definitions (3) in Equation (2), and the convolution property of characteristic functions (cf):

$$\Phi(u) = p \sum_{n=1}^{\infty} q^{n-1} \varphi^n(u) = p\varphi(u) \sum_{n=1}^{\infty} [q\varphi(u)]^{n-1}$$

$$= \frac{p\varphi(u)}{1 - q\varphi(u)} \quad , \tag{4}$$

where the indicated sum is valid because $q\varphi(u) < 1$, for all u. Equation (4) can now be inverted to give

$$h(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-iut} \Phi(u)du = \frac{p}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-iut} \varphi(u)du}{1 - q\varphi(u)} \quad . \tag{5}$$

We should note, in passing, that the result in Equation (4) can be written down at once after inspecting Equation (1). This comes about because in (1), the sequence of probabilities $p, pq, \ldots, pq^{n-1}, \ldots$ is the probability mass function (pmf) of the random variable N, the round number on which a hit occurs. It is geometric and given by

$$
\begin{aligned}
p_N(n) &= pq^{n-1}, \quad n = 1,2,\ldots, \\
&= 0 \quad , \text{ elsewhere} \quad ,
\end{aligned}
$$

which has a geometric transform

$$
\begin{aligned}
G_N(z) &= \sum_{n=-\infty}^{\infty} p_N(n) z^n = p \sum_{n=1}^{\infty} q^{n-1} z^n \\
&= pz \sum_{n=1}^{\infty} (qz)^{n-1} = \frac{pz}{1-qz} \quad . \qquad (6)
\end{aligned}
$$

There is a theorem (see, Giffin, 1975, Eqn. 4-28) which says that for a mixture of iid random variables:

$$
\Phi(u) = G_N[\varphi(u)] \quad . \qquad (7)
$$

Therefore, from (3) and (6), we have immediately $\Phi(u) = \frac{p\varphi(u)}{1 - q\varphi(u)}$, as before.

The solution to the duel may now be written down. If T_A and T_B are the random variables, times for A and B to hit a passive target, with pdf's $h_A(t)$ and $h_B(t)$, respectively, then the probability that A wins the duel, $P[A]$, is

$$
\begin{aligned}
P[A] &= P[T_A < T_B] \\
&= \int_0^\infty P[t < T_A < t + dt, T_B > t]dt \\
&= \int_0^\infty h_A(t)\left(\int_t^\infty h_B(\xi)d\xi\right)dt \\
&= \int_0^\infty h_A(t)\, H_B^c(t)dt \quad , \qquad (8)
\end{aligned}
$$

where the third line is justified by the independence of T_A and T_B.

This may be put in more tractable form using characteristic functions by a theorem of Parseval which states,

$$\int_{-\infty}^{\infty} f_1(x)\, f_2(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_1(-u)\, \varphi_2(u)du \quad . \qquad (9)$$

Using (9) in (8),

$$P[A] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_A(-u)\, \frac{[\Phi_B(u) - 1]}{iu}\, du \quad . \qquad (10)$$

It is easily shown that the $\Phi(-u)$ function has no poles in the complex lower half-plane and that it vanishes on a large semicircle, C, in the lower half-plane as $R \to \infty$ (see Fig. 1).

Thus, if we go to complex u, by analytic continuation, the second term in the integrand is zero and we have

$$P[A] = \frac{1}{2\pi i} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{\Phi_A(-u)\ \Phi_B(u)du}{u} , \tag{11}$$

where ε is less than the distance to the nearest pole in the lower half of the complex u plane. Since a draw is impossible with continuous firing times, $P[B] = 1 - P[A]$.

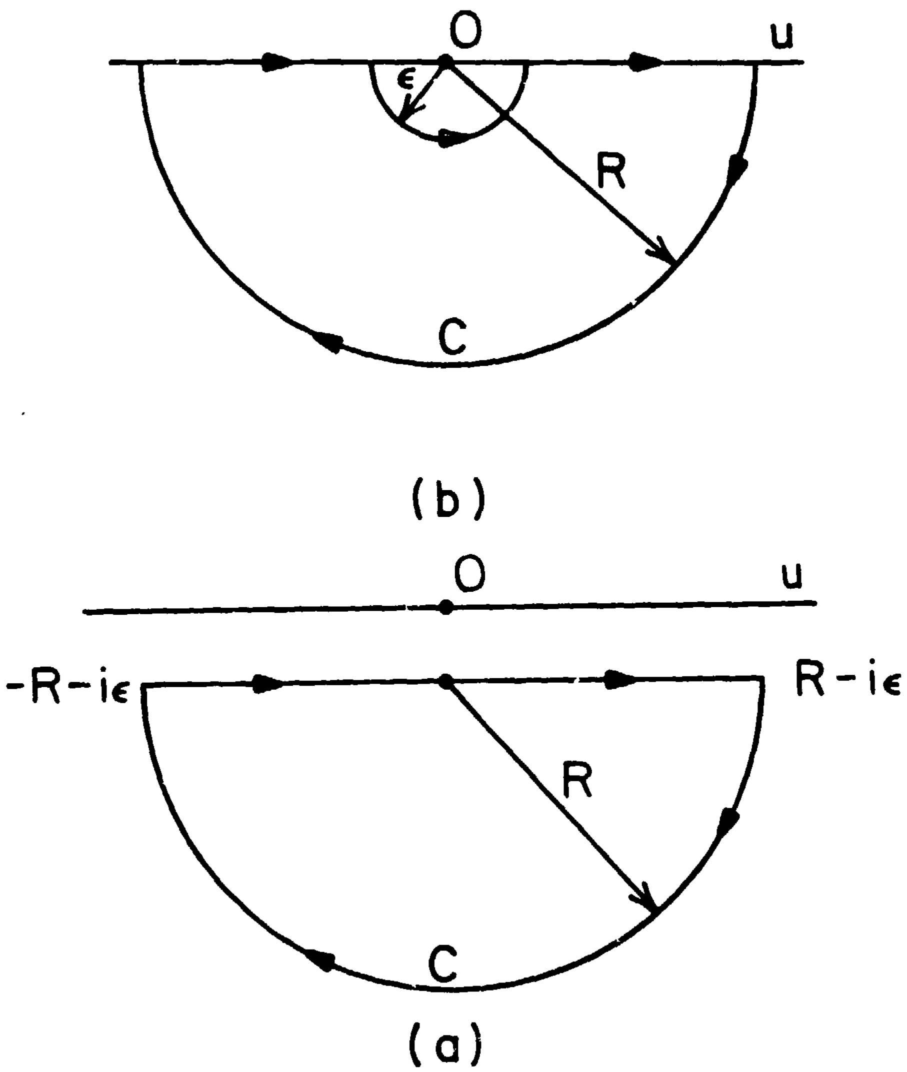

FIGURE 1

Path of Integration for $P[A]$ in Equation (11); also, frequently given as $\int_L$.

Much of the literature uses the Laplace transform (LT) of the time functions involved, but we shall always use the characteristic function for two reasons:

(1) the cf always exists, whereas the LT may not, and

(2) the numerical integration of the Fourier transform (cf) has received extensive attention and many efficient algorithms are available (see, for example, Brigham, 1974).

Finally, let us emphasize the efficiency and economy of using characteristic functions by comparing Equation (1) with the solution to the duel without their use. Putting (2) with appropriate subscripts for A and B into (8), we have

$$P[A] = \int_0^\infty \sum_{n=1}^{\infty} p_A q_A^{n-1} f_A^{n*}(t) \left(\int_t^\infty \sum_{n=1}^{\infty} p_B q_B^{n-1} f_B^{n*}(\xi) d\xi \right) dt , \quad (12)$$

and from (11),

$$P[A] = \frac{1}{2\pi i} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{p_A \varphi_A(-u)}{[1 - q_A \varphi_A(-u)]} \frac{p_B \varphi_B(u)}{[1 - q_B \varphi_B(u)]} \frac{du}{u} . \quad (13)$$

We see that in (13) we have one integration to perform, whereas in (12), we have two integrations of two infinite sums, each term of which involves iterated convolutions. This is indeed an enormous simplification.

2. SOLUTION METHODS

In this section we shall examine several solution techniques commonly used in the literature. For this purpose, let us solve the FM problem, slightly modified as follows. A marksman fires with constant hit probability, p, at a passive target. His target is destroyed when he hits it twice. His interfiring times are continuous, iid, and have a general (unspecified) pdf.

A. The Mixture and Characteristic Function Technique

Using the notation given above and proceeding as before,

$$\left.\begin{array}{lll} T = X_1 + X_2 & \text{with probability} & \dfrac{p_N(n)}{p^2} \\ \quad = X_1 + X_2 + X_3 & \text{"} & pqp + qpp \\ \quad = X_1 + X_2 + X_3 + X_4 & \text{"} & pqqp + qpqp + qqpp \\ \quad\vdots & \vdots & \vdots \\ \quad = X_1 + X_2 + \cdots + X_n & \text{"} & \binom{n-1}{1} p^2 q^{n-2} \\ \quad\vdots & \vdots & \vdots \end{array}\right\} . \tag{14}$$

Note that, in general, the n^{th} term has a coefficient $\binom{n-1}{1}$ because we can have the first hit anywhere in the first $n-1$ positions, while the second hit must be on the n^{th} firing.

Continuing as before,

$$h(t) = \sum_{n=2}^{\infty} \binom{n-1}{1} p^2 q^{n-2} f^{n*}(t) =$$

$$= \sum_{n=2}^{\infty} \binom{n-1}{n-2} p^2 q^{n-2} f^{n*}(t) ,$$

and taking the cf of both sides

$$\Phi(u) = \varphi^2(u)\, p^2 \sum_{n=2}^{\infty} \binom{n-1}{n-2} (q\,\varphi(u))^{n-2}$$

$$= [p\,\varphi(u)]^2 \sum_{n=0}^{\infty} \binom{n+2-1}{n} [q\,\varphi(u)]^n \quad . \qquad (15)$$

Recognizing the sum as the negative binomial series with parameter 2, we may immediately write

$$\Phi(u) = p^2 \varphi^2(u)[1 - q\,\varphi(u)]^{-2} = \left[\frac{p\,\varphi(u)}{1 - q\,\varphi(u)} \right]^2 . \qquad (16)$$

Two comments are in order here. First, the pmf for the round number (N) on which the process terminates is given by the right-hand column of (14) and is, of course, a negative binomial with parameter 2, identical to the one above. Its geometric transform is

$$G_N(z) = \left[\frac{pz}{1 - qz} \right]^2 ,$$

and using (7),

$$\Phi(u) = \left[\frac{p\,\varphi(u)}{1 - q\,\varphi(u)} \right]^2 ,$$

which is the same as (16). Secondly, we note that this result is obvious because the time to a second hit is simply the time to a first hit plus the independent time to a second hit, both of which are identically distributed as the time to a first hit whose cf is given by (4), and therefore, our result follows immediately. Although the result is obvious, it is still instructive to derive it by several different techniques.

B. The Semi-Markov Process Technique

In this technique, Markov chain theory is used to establish the pmf of N, the round number on which the contest terminates. This is, of course, the right-hand column of (14) and one might wonder why it would be advantageous to use a relatively complicated technique such as this to establish what appears to be rather obvious. The answer is two-fold:

(1) to demonstrate the technique, and

(2) when the state space is more complicated than in our example, the derivation can become very difficult as there will be so many branches in the outcome tree that keeping track of all the paths through it may become nearly impossible. The Markov chain does this automatically.

The first step is to establish the state space, which in our case, has three states: (1) not hit $(\bar{H})$, (2) hit but not killed $(H\bar{K})$, i.e., one hit only, and finally, (3) hit and killed, or simply killed (K), i.e., a second hit.

We notice that in the language of Markov chains, two states are transient $(H\bar{K}, \bar{H})$ and one (K) is absorbing (once entered, it may

never be left). Next, the initial state probability vector $(\underset{\sim}{I})$ and the transition probability matrix on the state space $(\underset{\sim}{S})$ are determined. In our example we have

$$\underset{\sim}{I} = \begin{array}{c} \begin{array}{ccc} K & H\bar{K} & \bar{H} \end{array} \\ (0\,,\,0\,,\,1) \end{array}$$

and

$$\underset{\sim}{S} = \begin{array}{c|ccc} & K & H\bar{K} & \bar{H} \\ \hline K & 1 & 0 & 0 \\ H\bar{K} & p & q & 0 \\ \bar{H} & 0 & p & q \end{array}$$

which are now partitioned as follows:

$$\underset{\sim}{I} = (0\,\vdots\,0\,,1) = (0\,,\underset{\sim}{M})$$

and

$$\underset{\sim}{S} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline p & q & 0 \\ 0 & p & q \end{array}\right) = \begin{pmatrix} 1 & 0 \\ \underset{\sim}{T} & \underset{\sim}{P} \end{pmatrix} .$$

The row vector $\underset{\sim}{M}$ excludes the absorbing state K. The sub-matrix $\underset{\sim}{P}$ contains all the transition probabilities for remaining in non-absorbing states and the column vector $\underset{\sim}{T}$ contains all the transition probabilities for going from a transient state ($H\bar{K}$ or $\bar{H}$) to the absorbing state K. From elementary Markov theory we have that $p_N(n)$, the probability of a

kill on the n^{th} round fired is

$$p_N(n) = \underset{\sim}{M}\,\underset{\sim}{P}^{n-1}\,\underset{\sim}{T} \ . \tag{17}$$

First, determine $\underset{\sim}{P}^{n-1}$ by matrix multiplication:

$$\underset{\sim}{P}^1 = \begin{pmatrix} q & 0 \\ p & q \end{pmatrix}$$

$$\underset{\sim}{P}^2 = \begin{pmatrix} q^2 & 0 \\ 2pq & q^2 \end{pmatrix}$$

$$\underset{\sim}{P}^3 = \begin{pmatrix} q^3 & 0 \\ 3pq^2 & q^3 \end{pmatrix}$$

$$\vdots$$

$$\underset{\sim}{P}^n = \begin{pmatrix} q^n & 0 \\ npq^{n-1} & q^n \end{pmatrix}$$

$$\vdots$$

The general result for $\underset{\sim}{P}^n$ is easily proven by induction. Continuing,

$$\underset{\sim}{M}\,\underset{\sim}{P}^{n-1} = \begin{pmatrix} 0\,, 1 \end{pmatrix}\begin{pmatrix} q^{n-1} & 0 \\ (n-1)pq^{n-2} & q^{n-1} \end{pmatrix}$$

$$= \left((n-1)pq^{n-2}\,, q^{n-1} \right)$$

and

$$p_N(n) = \underset{\sim}{M}\,\underset{\sim}{P}^{n-1}\,\underset{\sim}{T} = \left((n-1)pq^{n-2}, q^{n-1}\right)\binom{p}{0}$$

$$= (n-1)pq^{n-2} = \binom{n-1}{1}pq^{n-2} ,$$

which is, of course, the general term of the pmf of N and corresponds to the general term of the right-hand column of (14) above in A. From here, the development follows A above precisely and will not be repeated. The reason that this is called a semi-Markov process is because the time between events is a RV.

Before turning to the next topic, we should note that the state space may be much larger. There may be many transient states, thus, $\underset{\sim}{P}$, $\underset{\sim}{M}$ and $\underset{\sim}{T}$ may be much larger. In fact, there may be quite a few absorbing states. For example, it might be possible to be killed on the first hit, or not killed on the first hit and killed on the second hit (as above). The point is, that this technique provides a convenient and orderly way to expand the state space.

C. Renewal Theory Integral Equations Technique

It will be helpful in looking at the techniques to be described in what follows, to establish first some notation and concepts from renewal theory.

Let $N(t)$ be the number of firings up to time t. Clearly, for every fixed t, this is a discrete RV. The situation may be depicted graphically by a possible realization as shown in Figure 2. This process is called a terminating semi-Markov renewal process. Most of the renewal

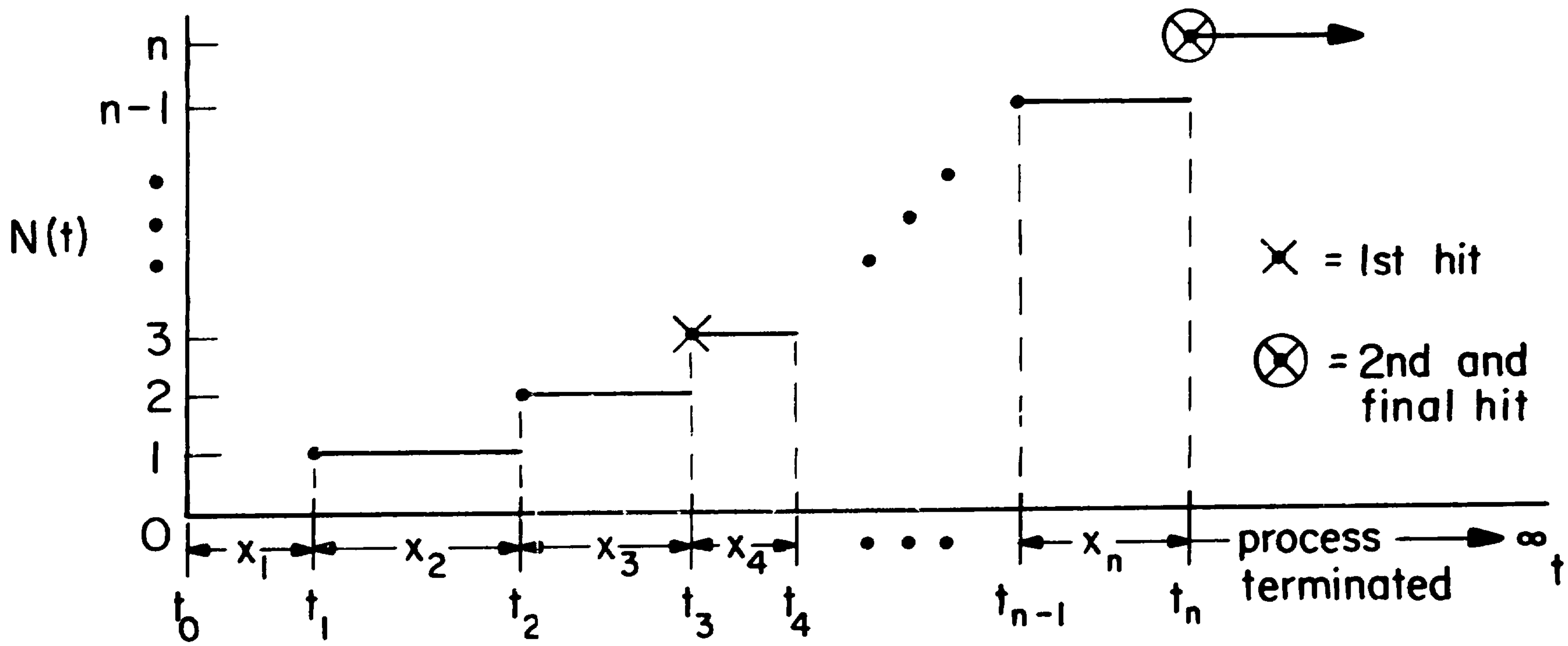

FIGURE 2

The Two-Hit, Terminating Semi-Markov Renewal Marksman Process

theory literature concerns itself with nonterminating processes, but we may still profitably consider this approach.

In the ordinary nonterminating renewal process, one important result is that $E[N(t)]$ can be expressed as a simple integral equation. In our case, however, we are more interested in the function $h(t)$ which can also, in a similar manner, be derived from a series of integral equations in sequential order. In order to do this we define three functions, as follows.

First, let $T_n = \Sigma_{i=1}^{n} X_i$ be the RV, time to n^{th} firing, with pdf $f^{n*}(t)$, then,

$$\begin{aligned} h_n^0(t)dt &= P[t < T_n < t+dt\,,\, 0 \text{ hits in } n \text{ trials}] \\ &= P[t < T_n < t+dt \mid 0 \text{ hits in } n \text{ trials}] \\ &\qquad \cdot P[0 \text{ hits in } n \text{ trials}] \\ &= f^{n*}(t)q^n\,dt, \qquad n = 1,2,\ldots, \end{aligned} \tag{18}$$

$$\begin{aligned} h_n^1(t)dt &= P[t < T_n < t+dt\,,\, 1 \text{ hit in } n \text{ trials}] \\ &= P[t < T_n < t+dt \mid 1 \text{ hit in } n \text{ trials}] \\ &\qquad \cdot P[1 \text{ hit in } n \text{ trials}] \\ &= f^{n*}(t)\binom{n}{1}pq^{n-1}\,dt, \qquad n = 1,2,\ldots, \end{aligned} \tag{19}$$

where the factor $\binom{n}{1}$ is necessary because the hit may occur on any round fired, and

$$h_n(t) = P[t < T_n < t + dt,\ 1 \text{ hit in } n-1 \text{ trials and } 2^{nd} \text{ hit on the } n^{th} \text{ trial}]$$

$$= P[t < T_n < t + dt \mid 1 \text{ hit in } n-1 \text{ trials and } 2^{nd} \text{ hit on the } n^{th} \text{ trial}]$$

$$\cdot\ P[1 \text{ hit in } n-1 \text{ trials and } 2^{nd} \text{ hit on the } n^{th} \text{ trial}]$$

$$= f^{n*}(t) \binom{n-1}{1} p^2 q^{n-2} dt, \qquad n = 2,3,\ldots, \quad . \tag{20}$$

Now, notice that if we consider the sequence $n = 1,2,\ldots,$ and consider the sum on n that (18) and (19) are not proper density functions, but that (20) is (it is just the mixture defined in (14)). What we are doing here is looking at the ensemble of all possible realizations of $N(t)$, such as Figure 2, and at every t selecting out all those which have had an event occur in $(t, t + dt)$. We then partition all these realizations into three subsets:

(1) those with no hits,

(2) those with only one hit, and

(3) those which have terminated.

We are, of course, looking at subsets of (1), (2), and (3) where $N = n$ for all possible values of N. With this understanding of the nature of $h_n^0(t)$, $h_n^1(t)$, and $h_n(t)$ we may now write three interconnected integral equations. For example,

$$h_n^0(t)dt = dt\, q \int_0^t f(x)\, h_{n-1}^0(t - x)dx \quad ,$$

which states that if no hits had been scored in $n-1$ rounds, the last of which was fired at time $t-x$, then if we fire another round at t it must fail, in order for the system to be in the state of having just fired a round which was the n^{th} one, and no hits have been made yet. The integral is just the convolution of $h^{o}_{n-1}(t)$ with $f(t)$. If we now define

$$
\begin{aligned}
h^{o}(t) &= \sum_{n=1}^{\infty} h^{o}_{n}(t) \\
&= q \int_0^t f(x) \sum_{n=1}^{\infty} h_{n-1}(t-x)dx \\
&= qf(t) + q \int_0^t f(x)\, h^{o}(t-x)dx \quad .
\end{aligned}
\tag{21}
$$

Note that for $n=1$ we merely have one selection from $f(t)$, multiplied by the probability of a failure, in order to get $h^{o}_{1}(t)$ (see Equation (18)).

In a similar manner,

$$
\begin{aligned}
&h^{1}_{n}(t)dt \\
&= dt\ p \int_0^t f(x)\, h^{o}_{n-1}(t-x)dx + dt\ q \int_0^t f(x)\, h^{1}_{n-1}(t-x)dx \quad ,
\end{aligned}
$$

where we have accounted for two mutually exclusive ways to get to the state of having fired one round. These are, no successful rounds in $n-1$ firings and a hit on the n^{th}, and one hit somewhere in the first $n-1$ rounds and no hit on the n^{th} round. Summing as before,

$h^1(t)$

$$= p f(t) + p \int_0^t f(x)\, h^0(t-x)dx + q \int_0^t f(x) h^1(t-x)dx \,, \tag{22}$$

where we note that $h_0^1(t) = 0$; and finally,

$$h_n(t)dt = dt\; p \int_0^t f(x)\, h_{n-1}^1(t-x)dx \quad ,$$

which, when summed gives,

$$h(t) = p \int_0^t f(x)\, h^1(t-x)dx \quad . \tag{23}$$

Equations (21), (22) and (23) may be solved in that order, as they depend on each other in that order. Taking the characteristic function of both sides of (21) and denoting the cf of $h^0(t)$ by $\Phi^0(u)$,

$$\Phi^0(u) = q\,\varphi(u) + q\,\varphi(u)\,\Phi^0(u) \quad ,$$

which, when solved for $\Phi^0(u)$ is

$$\Phi^0(u) = \frac{q\,\varphi(u)}{1 - q\,\varphi(u)} \quad . \tag{24}$$

Now, taking the cf of (22) and describing the cf of $h^1(t)$ as $\Phi^1(u)$,

$$\Phi^1(u) = p\,\varphi(u) + p\,\varphi(u)\,\Phi^0(u) + q\,\varphi(u)\,\Phi^1(u) \quad ,$$

substituting $\Phi^0(u)$ from (24) and solving for $\Phi^1(u)$, one obtains:

$$\Phi^1(u) = \frac{p\,\varphi(u)}{[1 - q\,\varphi(u)]^2} . \tag{25}$$

Finally, taking the cf of both sides of (23)

$$\Phi(u) = p\,\varphi(u)\,\Phi^1(u)$$

and substituting $\Phi^1(u)$ from (25) into this,

$$\Phi(u) = \left[\frac{p\,\varphi(u)}{1 - q\,\varphi(u)}\right]^2 , \tag{26}$$

which again is the same as (16), as it should be.

The advantage of this technique is that, in spite of the lengthy exposition given above, it is often the simplest and fastest way to a given desired result. Also, note that we have only used Equations (18), (19) and (20) for definitions. We could have proceeded directly to the solution with (20), as it is just the mixture solution. Also, we can use Equations (24) and (25) (or, (18) and (19)) to get additional information on the process. These have physical meanings such as: the inversion of (24) will give the (improper) density function of the time to zero hits, and (25) gives the pdf of the time to one hit. These additional relations may also be obtained by the mixture technique, but each would require a separate, independent calculation, in exactly the same manner as described for $h(t)$ in Section A above.

D. Renewal Theory Integro-Differential-Difference Equation Techniques

For the renewal process described in Section C above, the integro-differential-difference technique may be applied if the interfiring times have a negative exponential pdf (ned). The reason for this is that the ned RV has no memory and the process is reduced to a terminating semi-Markov renewal process as a result.

However, if the supplementary variable technique (see, Cox, 1955 or Keilson and Kooharian, 1960) is applied, the restriction to ned IFT's is removed and general IFT's may be considered at the cost of some complication in the mathematics. In this fashion, the process goes from non-Markov to semi-Markov. This procedure is widely used in the literature pertaining to Stochastic Duels, but much ambiguity, many notational difficulties and, in come cases, unnecessary complications are widespread. Consequently, we shall illustrate the method for our example in some detail, in the hope that the reader will find the literature more readily accessible.

As before, the process may be considered graphically as shown in Figure 3. The situation depicted there is for the case where the process is observed at some arbitrary time, t, and its state is noted. In this case, the state is that $N(t) = n$ and there have been no hits. The time since the last firing is a RV and is in the interval $(t-y, t-y+dy)$. The corresponding new random variable Y_n, is the supplementary variable and in renewal theory terms is the backward recurrence time (see, Cox and Miller, 1965, p. 339). The other two cases are:

(1) the same as shown in Figure 3, except that one hit has occured

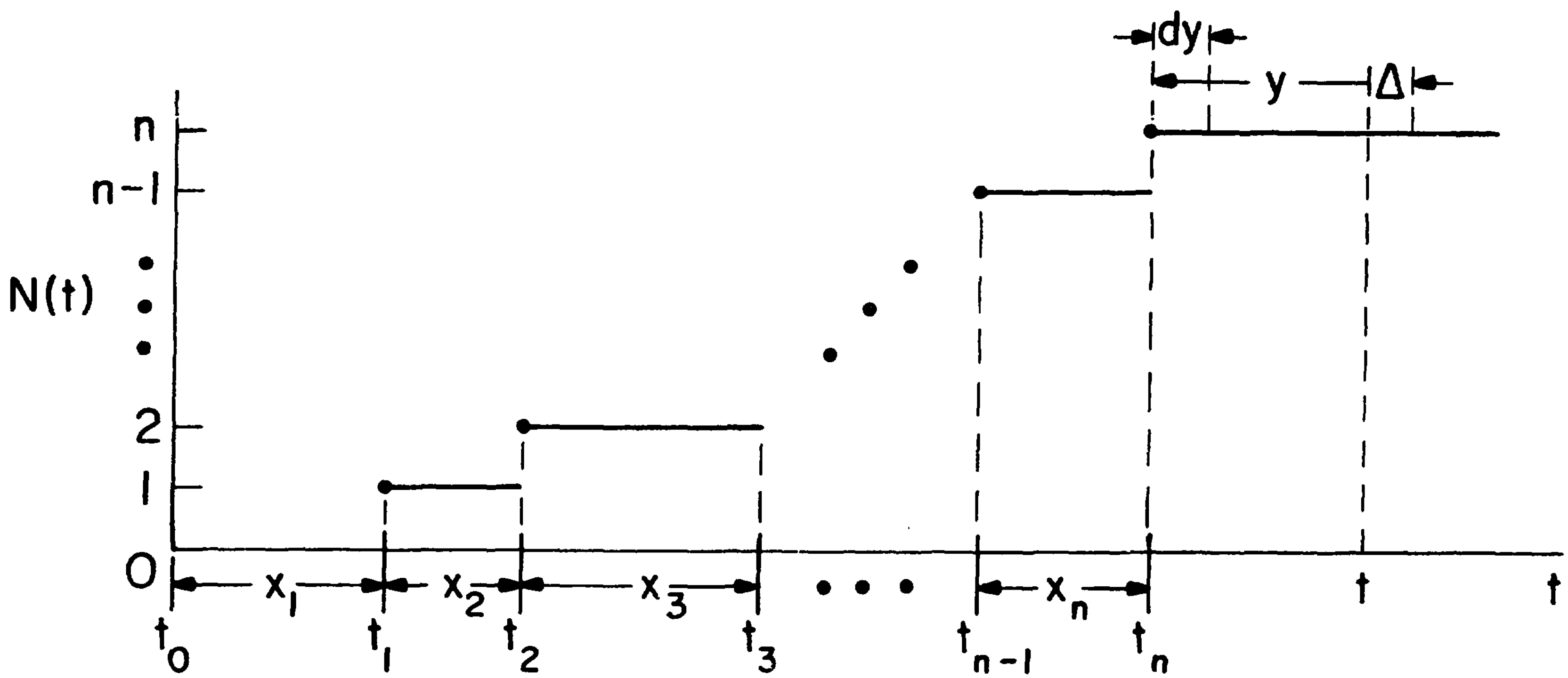

FIGURE 3

The Backward Recurrence Time

on one of the n firings (possibly, even on the n^{th} firing), and

(2) at the time of observation, the process has terminated, i.e., the second hit occurred on the n^{th} round.

In this latter case, the supplementary variable is unnecessary since no further events can take place. The supplementary variable is subscripted with n as it is a function of n, but its realization is not subscripted, since we wish to look backwards the same distance y for all Y_n's.

The properties of the supplementary variable, Y_n, and its relationships to the IFT, X (which is most important), are well known and may be summarized as follows:

(a) Y_n is measured to the left from t and $f_{Y_n}(y\,;t)dy$ is the probability that the n^{th} firing lies in the interval $(t-y\,,\,t-y+dy)$;

(b) $f_{Y_n}(y\,;t) = h_n^{0,1}(t-y)\,F_X^c(y),\quad 0 < y < t,$ where $h_n^{0,1}(\xi)$ is the already familiar pdf of the renewal process for either zero hits or one prior hit, with $N(t) = n$;

(c) $$P[y < X_{n+1} < y + dy \,|\, X_{n+1} > y] = P[y < X < y + dy \,|\, X > y]$$

$$= \lambda(y)dy = \frac{f_X(y)dy}{F_X^c(y)}\,,$$

which implies that the probability of a firing in the interval $(t\,,\,t+\Delta) = \lambda(y)\Delta + o(\Delta)$, where $\lim\limits_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$;

(d) $f_X(y) = \lambda(y)\, e^{-\int_0^y \lambda(\xi)d\xi}$; and

(e) from (c) and (d) above, $F_X^c(y) = e^{-\int_0^y \lambda(\xi)d\xi}$.

With this background, we now define a function which is the probability that, if the process is observed at some arbitrary time t, with n firings having occurred, and no hits have been made, and that the last firing was between y and $y - dy$ time units earlier.

$$H_n^o(t, y)dy = P[N^o(t) = n, y < Y_n < y - dy]$$

$$= \begin{cases} P[T_n^o < t, y < Y_n < y - dy]; & 0 < y < t,\ n > 0 \\ 0 & ;\quad y \geq t > 0,\ n > 0 \end{cases},$$

$$H_o^o(t, y)dy = \delta(y - t)\, F_X^c(y) \qquad ; \quad y = t > 0 \qquad . \qquad (27)$$

The first form of (27) is to be interpreted as a joint probability mass function (improper) on the number of rounds fired with no hits occurring ($N^o(t)$), and a probability density function on the time since the last firing, Y_n, with a parameter t. The second (equivalent) form is interpreted as a joint distribution function (improper) on T_n^o, the time of firing the n^{th} round with no hits and a pdf on Yn, with a parameter n. We also note that when $n = 0$, that y must equal t, which accounts for the Dirac delta function $\delta(y - t)$.

In an exactly similar fashion we have,

$$H_n^1(t, y)dy = P[N^1(t) = n, y < Y_n < y - dy]$$

$$= \begin{cases} P[T_n^1 < t, y < Y_n < y - dy]; & 0 < y < t, n \geq 1 \\ 0 & ; \quad y > t, n \geq 1 \end{cases}, \quad (28)$$

where $N^1(t)$ is the RV number of rounds fired with one hit somewhere, and where T_n^1 is the RV time of firing of the n^{th} round with one hit in one of the n rounds fired. Now, the case where the process has terminated is described by

$$H_n(t) = P[N(t) = n, \ 2^{nd} \text{ hit on the } n^{th} \text{ round}]$$

$$= P[T_n < t, \text{process terminated}]; \qquad t \geq 0, n \geq 2 \quad . \quad (29)$$

In this case, the first form is a proper pmf on N with parameter t and the second form is a proper df on Tn where n is a parameter.

The fact that certain of the mass and density functions above are improper will be seen as we derive them. As in Section C, above, this occurs whenever we consider a subset of the sample space which is not just the terminated subset. The reason for this is not easy to see, but it is related to the fact that all realizations terminate with probability one.

With this background we may now derive the integro-differential-difference equations which govern the process. If we ask about the state at some forward time, $t + \Delta$, note that this also simultaneously extends y to $y + \Delta$.

Now, from elementary calculus and referring to Figure 4, if we ask

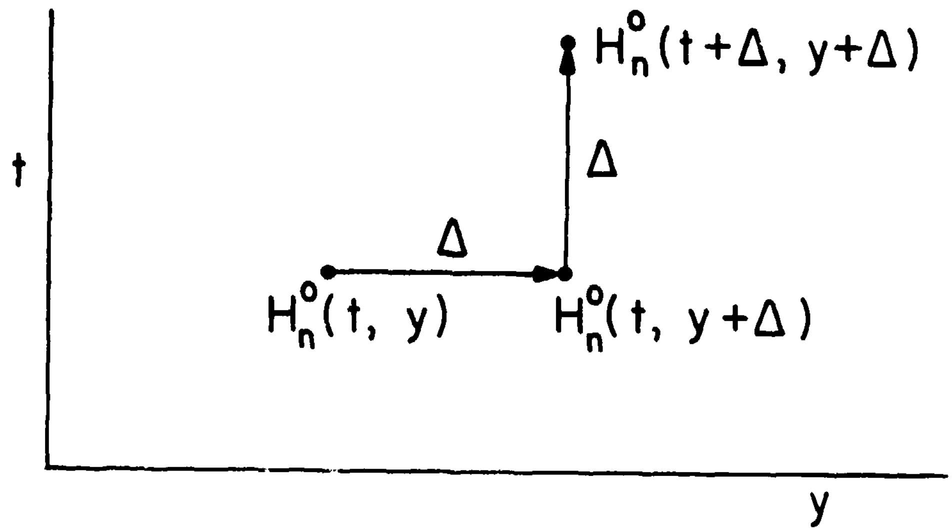

FIGURE 4

about being in state: n firings and no hits at $t+\Delta$, we have

$$H_n^o(t + \Delta , y + \Delta)$$
$$= H_n^o(t , y) + \frac{\partial}{\partial y} H_n^o(t , y)\Delta + \frac{\partial}{\partial t} H_n^o(t , y + \Delta)\Delta \quad .$$

Also, this same probability is that we are in the state: n firings and no hits at t, and no firing occurs in the interval $(t , t+\Delta)$, i.e., $H_n^o(t , y)[1 - \lambda(y)\Delta]$. Equating these two statements, we have,

$$H_n^o(t , y)[1 - \lambda(y)\Delta]$$
$$= H_n^o(t , y) + \frac{\partial}{\partial y} H_n^o(t , y)\Delta + \frac{\partial}{\partial t} H_n^o(t , y + \Delta)\Delta \quad .$$

Rearranging terms, dividing through by Δ, and taking the limit as $\Delta \to 0$ gives,

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \lambda(y) \right) H_n^0(t, y) = 0; \quad n = 0,1,2,\ldots, \ 0 < y < t. \tag{30}$$

Note that, although $H_0^0(t, y)$ is quite different in form from $H_n^0(t, y)$ for $n > 0$, it still satisfies (30).

In a precisely similar manner, we can immediately write down for the state: n firings and one hit,

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \lambda(y) \right) H_n^1(t, y) = 0; \quad n = 1,2,\ldots, \ 0 < y < t. \tag{31}$$

For the situation of n firings and two hits (process terminated), there are two mutually exclusive possibilities when we go to $t+\Delta$ from t. Either the process had terminated before t (and therefore, remains terminated), or it had not terminated earlier and does terminate in $(t, t+\Delta)$, thus,

$$H_n(t + \Delta) = H_n(t) + \int_0^t H_{n-1}^1(t, y)\, \lambda(y) \Delta p \, dy \ ;$$

$$n \geq 2, \ t > 0 \ .$$

Rearranging terms, dividing through by Δ and taking the limit as $\Delta \to 0$,

$$\frac{d}{dt}\left[H_n(t) \right] = h_n(t) = p \int_0^t H_{n-1}^1(t, y)\, \lambda(y) dy \ ;$$

$$n \geq 2, \ t > 0 \ . \tag{32}$$

Two boundary conditions must now be accounted for, as follows. Define

$$h_n^o(t)dt = P[t < T_n^o < t + dt] \tag{33}$$

from which

$$h_n^o(t)\Delta = \int_0^t H_{n-1}^o(t, y)\, \lambda(y)\Delta q\, dy \quad , \tag{34}$$

where the right-hand side of (34) is the probability of being, at time t, in state $n - 1$ firings, with all failures and time y since the last firing and a firing in the interval $(t, t + \Delta)$ with a failure at that firing, integrated over all permissible values of y. From (34)

$$h_n^o(t) = q \int_0^t H_{n-1}^o(t, y)\, \lambda(y)dy, \qquad n \geq 1 \quad . \tag{35}$$

Also,

$$h_o^o(t) = 0 \quad , \tag{36}$$

since $H_{-1}^o(t, y)$ is undefined. Equivalently, we take the time of firing the zeroeth round to be at $t = 0$.

By similar reasoning, we obtain

$$h_n^1(t)$$

$$= q \int_0^t H_{n-1}^1(t, y)\, \lambda(y)dy + p \int_0^t H_{n-1}^o(t, y)\, \lambda(y)dy, \quad n \geq 1 \; , \tag{37}$$

and

$$h_o^1(t) = 0 \quad , \tag{38}$$

since there is no way to have a hit without a firing. The two terms on the right-hand side of (37) account for the two ways to get to one hit at exactly time t, i.e., no prior hits and get a hit, and secondly, have a prior hit and get a miss at t.

We now define the following functions

$$\left.\begin{aligned} H^o(t, y) &= \sum_{n=0}^{\infty} H_n^o(t, y) \\ H^1(t, y) &= \sum_{n=1}^{\infty} H_n^1(t, y) \\ h(t) &= \sum_{n=2}^{\infty} h_n(t) \\ h^o(t) &= \sum_{n=1}^{\infty} h_n^o(t) \\ h^1(t) &= \sum_{n=1}^{\infty} h_n^1(t) \end{aligned}\right\} . \tag{39}$$

Note that upon summing on n,

$$H^o(t, y)dy = P[T^o < t, y < Y < y - dy] ,$$

where T^o is the RV, time to a firing (for <u>any</u> n) with no hits and Y is the RV, time since last firing (for any n). Similarly, $H^1(t,y)$ involves T^1 the RV, time to a firing (for any n) with one prior hit. Another important point is that

$$\lim_{y \to 0} H^o(t, y)dt = P[T^o < t, 0 < Y < 0 - dy]$$

$$= P[T < T^o < t + dt]$$

$$= \delta(-t)dt + h^o(t)dt \quad , \qquad (40)$$

from Equations (27) and (39). This is because the definition of $H_n^o(t, y)$ includes the case of $n = 0$ and $h_n^o(t)$ does not. However,

$$\lim_{y \to 0} H^1(t, y) = h^1(t) \quad . \qquad (41)$$

Now, upon summing on n and applying definitions (39) to Equations (30), (31), (32), (35) and (37),

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \lambda(y) \right) H^o(t, y) = 0, \qquad 0 < y < t, \qquad (42)$$

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \lambda(y) \right) H^1(t, y) = 0, \qquad 0 < y < t, \qquad (43)$$

$$h(t) = p \int_0^t H^1(t,y)\, \lambda(y)dy \quad , \qquad t > 0, \qquad (44)$$

$$h^o(t) = q \int_0^t H^o(t, y)\, \lambda(y)dy \quad , \qquad t > 0, \qquad (45)$$

$$h^1(t) = q \int_0^t H^1(t, y)\, \lambda(y)dy + p \int_0^t H^o(t, y)\, \lambda(y)dy \;, \; t > 0, \qquad (46)$$

and finally, the initial condition from (27),

$$H^o(0, y) = \delta(y) \quad . \qquad (47)$$

This comes about because, when $t = 0$, $y = 0$ and $F_X^c(0) = 1$.

Before proceeding, it must be noted that, in most of the literature, the preceding step is not a simple summation on n, but rather, the formation of a geometric transformation (sometimes referred to as a probability generating function or Z transform) first. This is done, by say, $\Sigma_{n=0}^{\infty} z^n H_n^o(t, y)$ where z is the transform variable. In almost all cases, this step is unnecessary and yields nothing, since authors usually set $Z = 1$, and we are right where we are now. The only reason for using the Z transform technique at this point is if one wishes to discover some of the properties of the pmf on $N(t)$. This may be done in the usual fashion by taking appropriate derivatives of the Z transform with respect to z.

The next step is to convert all functions to their characteristic functions, in the variable t, by the appropriate transformation. We define:

$$\text{cf's:} \qquad \left.\begin{array}{lcl} H^o(t, y) & \sim & \Psi^o(u, y) \\ H^1(t, y) & \sim & \Psi^1(u, y) \\ h(t) & \sim & \Phi(u) \\ h^o(t) & \sim & \Phi^o(u) \\ h^1(t) & \sim & \Phi^1(u) \end{array}\right\} \quad . \tag{48}$$

In performing these transforms, we note that the

$$\text{cf}\left(\frac{d}{dt} H^o(t, y) \right) = -iu\, \Psi^1(u, y) - H^o(0, y) \quad .$$

Proceeding to operate on Equations (42) through (46) and using (47),

$$\left(\frac{\partial}{\partial y} - iu + \lambda(y) \right) \Psi^0(u, y) = \delta(y) , \qquad y > 0 , \tag{49}$$

$$\left(\frac{\partial}{\partial y} - iu + \lambda(y) \right) \Psi^1(u, y) = 0 , \qquad y > 0 , \tag{50}$$

$$\Phi(u) = p \int_0^\infty \Psi^1(u, y) \, \lambda(y) dy , \tag{51}$$

$$\Phi^0(u) = q \int_0^\infty \Psi^0(u, y) \, \lambda(y) dy , \tag{52}$$

$$\Phi^1(u) = q \int_0^\infty \Psi^1(u, y) \, \lambda(y) dy + p \int_0^\infty \Psi^0(u, y) \, \lambda(y) dy . \tag{53}$$

We shall illustrate how the right-hand side (rhs) of the last three equations come about by looking at the derivation of the rhs of (51).

$$\begin{aligned} \text{rhs} &= p \int_0^\infty e^{iut} \, dt \int_0^t H^1(t, y) \, \lambda(y) dy \\ &= p \int_0^\infty dt \int_0^t H^1(t, y) e^{iut} \, \lambda(y) dy . \end{aligned}$$

Now, since $y \leq t$, the integration above is over the shaded region depicted in Figure 5, first in the y direction and then in the t direction. Reversing the order of integration we have

$$\text{rhs} = p \int_0^\infty \lambda(y) dy \int_y^\infty e^{iut} \, H^1(t, y) dt ,$$

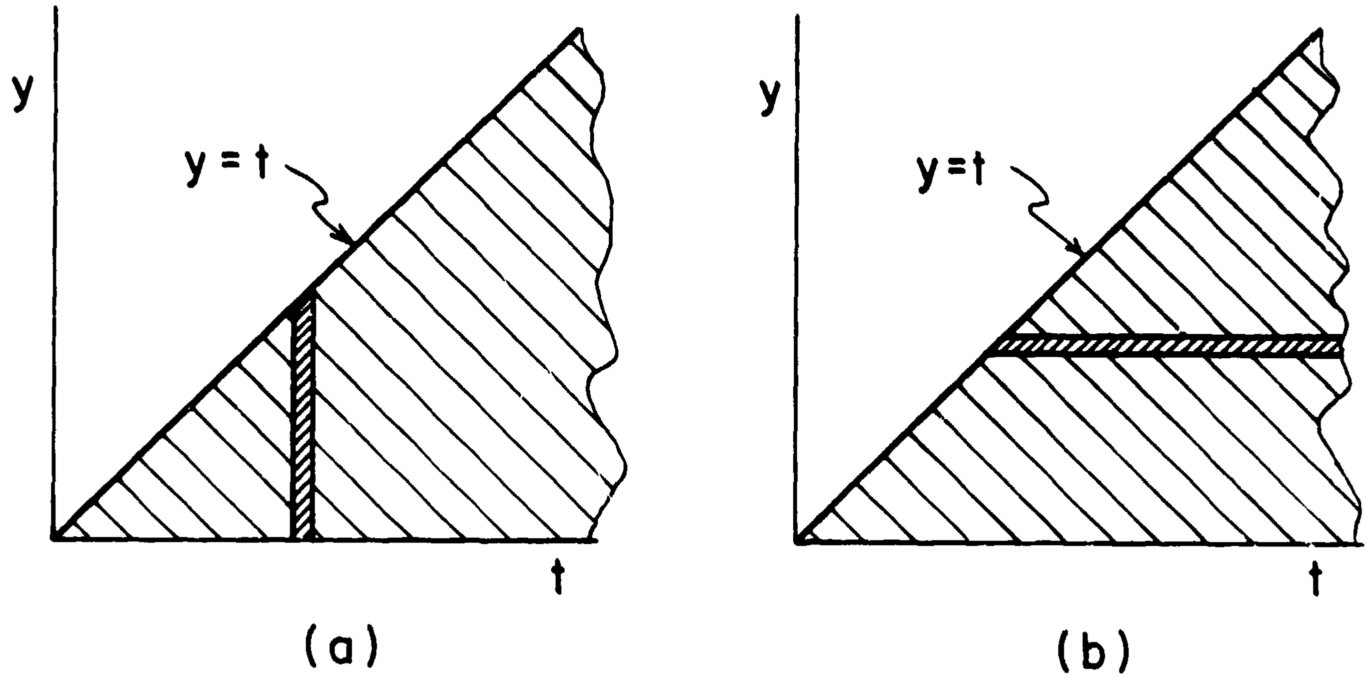

FIGURE 5

Integrating to Obtain Equation (51)

and since $H^1(t, y) = 0$ for $t < y$, the inner integral is $\Psi^1(u, y)$ and we have the desired result.

The problem now is to solve the simultaneous set of equations (49) through (53). Equation (49) is a linear, nonhomogeneous differential equation in the variable y. This may be solved in the usual way by getting the general solution to the homogeneous equation (rhs = 0) and adding a particular solution for the entire equation. Thus, setting rhs = 0,

$$\frac{\frac{\partial}{\partial y} \Psi_g^o(u, y)}{\Psi_g^o(u, y)} = iu - \lambda(y) \quad , \quad \text{or} \tag{54a}$$

$$\frac{\partial \ln \Psi_g^o(u, y)}{\partial y} = iu - \lambda(y) \quad , \quad \text{or} \tag{54b}$$

$$\ln \Psi_g^o(u,y) = iuy - \int_0^y \lambda(\xi)d\xi + c \quad , \tag{54c}$$

or
$$\Psi_g^o(u,y) = K\, e^{iuy - \int_0^y \lambda(\xi)d\xi} \tag{54d}$$

where K and c are some constants, and Ψ_g^o is the general solution to the homogeneous equation.

Now, let us try the following particular solution, Ψ_p^o,

$$\Psi_p^o(u,y) = U(y)e^{iuy - \int_0^y \lambda(\xi)d\xi} \tag{55}$$

where $U(y)$ is the unit step function, in (49), and we see at once that it is satisfied. Therefore,

$$\Psi^o(u,y) = \Psi_g^o + \Psi_p^o = [U(y) + K]e^{iuy - \int_0^y \lambda(\xi)d\xi} \quad . \tag{56}$$

We notice, from (40) and (48), that if we let $y \to 0$,

$$\Psi^o(u) = 1 + \Phi^o(u) \quad ,$$

which immediately gives us that $K = \Phi^o(u)$. Thus,

$$\Psi^o(u,y) = [U(y) + \Phi^o(u)]e^{iuy - \int_0^y \lambda(\xi)d\xi} \quad . \tag{57}$$

The solution to (50) follows immediately, since there is no complicating rhs and we may write

$$\Psi^1(u, y) = \Phi^1(u)\, e^{iuy - \int_0^y \lambda(\xi) d\xi} . \tag{58}$$

Now, substituting (57) into (52),

$$\Phi^0(u) = q \int_0^\infty [U(y) + \Phi^0(u)] e^{iuy - \int_0^y \lambda(\xi) d\xi} \lambda(y) dy . \tag{59}$$

Remembering that $U(y) = 1$ for $y > 0$, the integral on the rhs is just the cf of the pdf of X. Therefore, (59) is

$$\Phi^0(u) = q[1 + \Phi^0(u)]\, \varphi(u) ,$$

from which

$$\Phi^0(u) = \frac{q\,\varphi(u)}{1 - q\,\varphi(u)} . \tag{60}$$

Substituting (60) into (57)

$$\Psi^0(u, y) = \left[U(y) + \frac{q\,\varphi(u)}{1 - q\,\varphi(u)} \right] e^{iuy - \int_0^y \lambda(\xi) d\xi} . \tag{61}$$

Substituting Equations (58) and (61) into (53)

$$\Phi^1(u) = q\,\Phi^1(u) \int_0^{\infty} e^{iuy - \int_0^y \lambda(\xi)d\xi}\,\lambda(y)dy$$

$$+\; p \int_0^{\infty} \left[U(y) + \frac{q\,\varphi(u)}{1 - q\,\varphi(u)} \right] e^{iuy - \int_0^y \lambda(\xi)d\xi}\,\lambda(y)dy \quad ,$$

or

$$\Phi^1(u) \;=\; q\,\Phi^1(u)\;\varphi(u) + p\,\varphi(u)\left[1 + \frac{q\varphi(u)}{1 - q\,\varphi(u)} \right] \quad ,$$

or

$$\Phi^1(u) \;=\; \frac{p\varphi(u)}{[1 - q\,\varphi(u)]^2} \quad , \tag{62}$$

which, for (58) gives

$$\Psi^1(u\,,\,y) \;=\; \frac{p\varphi(u)}{[1 - q\,\varphi(u)]^2}\; e^{iuy - \int_0^y \lambda(\xi)d\xi} \quad . \tag{63}$$

Finally, (63) is substituted into (51)

$$\Phi(u) \;=\; \frac{p^2\,\varphi(u)}{[1 - q\,\varphi(u)]^2} \int_0^{\infty} e^{iuy - \int_0^y \lambda(\xi)d\xi}\,\lambda(y)dy \quad ,$$

or

$$\Phi(u) \;=\; \left[\frac{p\,\varphi(u)}{1 - q\,\varphi(u)} \right]^2 \quad , \tag{64}$$

which is the desired result and checks with all previous derivations.

At this point, one might question the need to use such an involved and complicated method to get the result in Equation (64) when other methods are simpler. The answer to this is that if it is the only result desired, then there is no advantage. However, it should be noted that the reward for the additional effort is, as it should be, that additional information is available, namely, one can invert Equations (60), (61), (62), and (63) to get probability functions on the distribution of times for zero hits, one hit, joint times for zero hits and time since last hit, and one hit and time since last hit. Also, as previously noted, if the geometric transform is used, information on the number of rounds fired can be obtained.

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2. ANCKER, C.J., Jr. "The Status of Developments in the Theory of Stochastic Duels - II," Operations Research, Vol. 15, No. 3, May-June, 1967, pp. 388-406.

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6. THOMPSON, D. See, T1 in Research Papers, pp. 573-576.

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APPENDIX

SOME USEFUL RESULTS IN THE THEORY OF CHARACTERISTIC FUNCTIONS

0. Introduction

This appendix contains some theorems and other results from the theory of characteristic functions which are useful in derivations or applications in the Theory of Stochastic Duels. All of these results, except Theorem (A15), are from a compilation in bibliographic item A6. The proof of Theorem (A15) is given herein.

We use $\varphi_X(u)$ to denote the characteristic function of the pdf $f_X(t)$. Except for the Parseval theorems, the results are for positive RV's only. The notations $\int_L$ and $\int_U$ are the same as those given earlier in this work.

1. Some Parseval Theorems

Three useful versions of Parseval's Theorem are given below.

$$\int_{-\infty}^{\infty} f_A(t)\, f_B(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_A(-u)\, \varphi_B(u)du \quad , \tag{A1}$$

$$\int_{-\infty}^{\infty} e^{iut}\, f_A(t)\, f_B(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_A(u-w)\, \varphi_B(w)dw \quad , \tag{A2}$$

$$\int_{-\infty}^{\infty} e^{iut}\, f_A(t)\, f_B(t)\, f_C(t)dt$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \varphi_A(u-w)\Big(\int_{-\infty}^{\infty} \varphi_B(w-v)\, \varphi_C(v)dv\Big)dw. \tag{A3}$$

2. Properties of Characteristic Functions of Positive Random Variables

In what follows, only positive random variables are considered, i.e., pdf's such that

$$\left.\begin{array}{ll} f(t) \geq 0, & t \geq 0 \\ f(t) = 0, & t < 0 \end{array}\right\} \tag{A4a}$$

and

$$\int_0^\infty f(t)dt = 1 \tag{A4b}$$

with characteristic function and inverse

$$\left.\begin{array}{l} \varphi(u) = \int_0^\infty e^{iut} f(t)dt \\ f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-iut} \varphi(u)du \end{array}\right\} u \text{ real,} \tag{A5}$$

respectively.

The properties of interest follow.

$$\varphi(0) = 1 \quad , \tag{A6}$$

$$|\varphi(u)| \leq 1, \quad \text{Imaginary } u > 0 \quad . \tag{A7}$$

This implies no singularities in the upper half of the complex plane.

$$|\varphi(-u)| \leq 1, \quad \text{Imaginary } u \leq 0 \ . \tag{A8}$$

This implies no singularities in the lower half of the complex plane.

$$|\varphi(u)| \leq \frac{k}{|u|}, \quad \begin{array}{l} f(t) \text{ a differentiable function} \\ \text{of bounded variation. Imaginary} \\ u \geq 0, \ k = \text{positive constant.} \end{array} \tag{A9}$$

This implies that $\varphi(u)$ diminishes as $1/R$ in the upper half-plane where R is the radius of a semicircular path of integration in the complex plane.

$$|\varphi(-u)| \leq \frac{k}{|u|}, \quad \begin{array}{l} f(t) \text{ a differentiable function} \\ \text{of bounded variation. Imaginary} \\ u \leq 0, \ k = \text{positive constant.} \end{array} \tag{A10}$$

This implies that $\varphi(-u)$ diminishes as $1/R$ in the lower half-plane, R the same as in (A9).

$$|\varphi(u-w)| \leq 1, \quad \left\{ \begin{array}{l} \text{Imaginary } u \geq 0 \\ \text{Imaginary } w \leq 0 \end{array} \right. \ . \tag{A11}$$

This implies no singularities in the upper half of the complex u plane and the lower half of the complex w plane.

$$|\varphi(u-w)| \leq \frac{k}{|u-w|}, \quad \begin{array}{l} f(t) \text{ a differentiable function} \\ \text{of bounded variation. Imaginary} \\ u \geq 0, \ \text{imaginary } w \leq 0, \ \text{and} \\ k = \text{positive constant.} \end{array} \tag{A12}$$

This implies that $\varphi(u - w)$ diminishes as $1/R$ in the upper half of the u plane and in the lower half of the w plane. R, as above.

3. Some Theorems Involving Characteristic Functions of Positive Random Variables

We have that

$$\int_0^a e^{iut} f(t)dt = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(w + u)(1 - e^{-iwa})}{w} dw . \qquad (A13)$$

Note that for $u = 0$, this also gives an expression for the distribution function of a random variable in terms of characteristic functions. Also,

$$\int_a^{\infty} e^{iut} f(t)dt = \varphi(u) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(w + u)(1 - e^{-iwa})}{w} dw . \qquad (A14)$$

Note that for $u = 0$, this provides an expression for the complementary distribution function in terms of characteristic functions.

The characteristic function of the distribution function is given as:

$$\int_0^{\infty} e^{iut} \left(\int_0^t f(\xi)d\xi \right) dt = - \frac{\varphi(u)}{iu}, \qquad \text{Imaginary } u > 0 . \qquad (A15)$$

Proof: Integrating by parts,

$$\int_0^{\infty} e^{iut} \left(\int_0^t f(\xi)d\xi \right)dt$$

$$= \frac{e^{iut}}{iu} \int_0^t f(\xi)d\xi \Big|_{t=0}^{\infty} - \int_0^{\infty} \frac{e^{iut}}{iu} f(t)dt$$

$$= \lim_{t \to \infty} \frac{e^{iut}}{iu} - \frac{\varphi(u)}{iu} .$$

Now, to explore the first term, let u be analytically continued into the complex plane. Thus, $u = x + iy$, x,y real.

$$\frac{1}{iu} \lim_{t\to\infty} e^{iut} = \frac{1}{iu} \lim_{t\to\infty} e^{ixt} e^{-yt} .$$

Since $|e^{ixt}| = 1$,

$$\frac{1}{iu} \lim_{t\to\infty} e^{iut} \longrightarrow \begin{cases} 0 & , \; y > 0, \text{ any } x \\ \text{indeterminate}, & y = 0, \text{ any } x \\ \infty & , \; y < 0, \text{ any } x \end{cases} .$$

Q.E.D.

Continuing our listing of results,

$$\int_0^\infty e^{iut} \left(\int_t^\infty f(\xi)d\xi \right) dt = \frac{\varphi(u) - 1}{iu} . \tag{A16}$$

This is the characteristic function of the complementary distribution function.

$$\int_0^\infty e^{iut} f_A(t) \left(\int_t^\infty f_B(\xi)d\xi \right) dt = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\varphi_A(u-w)[\varphi_B(w) - 1]}{w} dw , \tag{A17}$$

$$\int_0^\infty f_A(t) \left[\int_t^\infty f_B(\xi)d\xi \right] dt = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\varphi_A(-u)[\varphi_B(u) - 1]}{u} du , \tag{A18}$$

$$\int_0^\infty f_A(t)\left[\int_t^\infty f_B(\xi)d\xi\right]\left[\int_t^\infty f_C(\eta)d\eta\right] dt$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{[\varphi_C(-u) - 1]}{u}\left(\int_{-\infty}^{\infty} \frac{\varphi_A(u-w)[\varphi_B(w)-1]}{w}\, dw\right) du\ , \qquad \text{(A19)}$$

$$\int_0^\tau f_A(t)\left(\int_t^\infty f_B(\xi)d\xi\right) dt$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{[e^{-iu\tau} - 1]}{u}\left(\int_{-\infty}^{\infty} \frac{\varphi_A(u - w)[\varphi_B(w) - 1]}{w}\, dw\right) du\ , \qquad \text{(A20)}$$

$$\int_\tau^\infty f_A(t)\left(\int_t^\infty f_B(\xi)d\xi\right) dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi_A(-u)[\varphi_B(u) - 1]}{u}\, du - \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{[e^{-iu\tau} - 1]}{u}$$

$$\cdot\left(\int_{-\infty}^{\infty} \frac{\varphi_A(u - w)[\varphi_B(w) - 1]}{w}\, dw\right) du\ , \qquad \text{(A21)}$$

$$\int_0^\infty f_A(t)\left(\int_{t+a}^\infty f_B(\xi)d\xi\right) dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-iau}\,\varphi_A(-u)[\varphi_B(u) - 1]}{u}\, du\ , \qquad \text{(A22)}$$

$$\int_0^\infty f_A(t)\left(\int_{at+b}^\infty f_B(\xi)d\xi\right)dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^\infty e^{-iub}\,\varphi_A(-au)\,\frac{[\varphi_B(u)-1]}{u}\,du \quad , \qquad \text{(A23)}$$

$$\int_0^\tau f_A(t)\left(\int_{at+b}^\infty f_B(\xi)d\xi\right)dt$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^\infty \frac{[e^{-iu\tau}-1]}{u}\left(\int_{-\infty}^\infty \frac{\varphi_A(u-aw)\,e^{-ibw}\,[\varphi_B(w)-1]}{w}\,dw\right)du \quad , \qquad \text{(A24)}$$

$$\int_\tau^\infty f_A(t)\left(\int_{at+b}^\infty f_B(\xi)d\xi\right)dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{e^{-iub}\,\varphi_A(-au)[\varphi_B(u)-1]}{u}\,du$$

$$- \frac{1}{4\pi^2}\int_{-\infty}^\infty \frac{[e^{-iu\tau}-1]}{u}\left(\int_{-\infty}^\infty \frac{\varphi_A(u-aw)\,e^{-iwb}\,[\varphi_B(w)-1]}{w}\,dw\right)du \; ; \qquad \text{(A25)}$$

$$\int_0^\infty f_A(t)\left(\int_t^\infty f_B(\xi)d\xi\right)\left(\int_t^\infty f_C(\eta)\left[\int_\eta^\infty f_D(\rho)d\rho\right]d\eta\right)dt \; ,$$

$$= \frac{1}{8\pi^3 i}\int_{-\infty}^\infty \frac{1}{u}\left(\int_{-\infty}^\infty \frac{\varphi_C(-u-v)[\varphi_D(v)-1]dv}{v} - 1\right)$$

$$\cdot\left(\int_{-\infty}^\infty \frac{\varphi_A(u-w)[\varphi_B(w)-1]dw}{w}\right)du \quad . \qquad \text{(A26)}$$

Our final result is that for any integral of the form

$$P[A] = \frac{1}{2\pi i} \int_L \frac{\varphi_A(-u)\ \varphi_B(u)}{u}\ du \ ,$$

where there may be any number of cf's of positive RV's in the integrand, as long as at least one has a positive argument and at least one has a negative argument and all are divided by u, then

$$P[A] = 2 \int_C \text{Re Integrand} \tag{A27}$$

where C is any path in the lower right half of the complex u plane, which starts at $u = 0$ and terminates at $+\infty$. The path must remain on the real u axis or be between the real u axis and the nearest singularity in the lower half plane. Examples are shown in Figure A1.

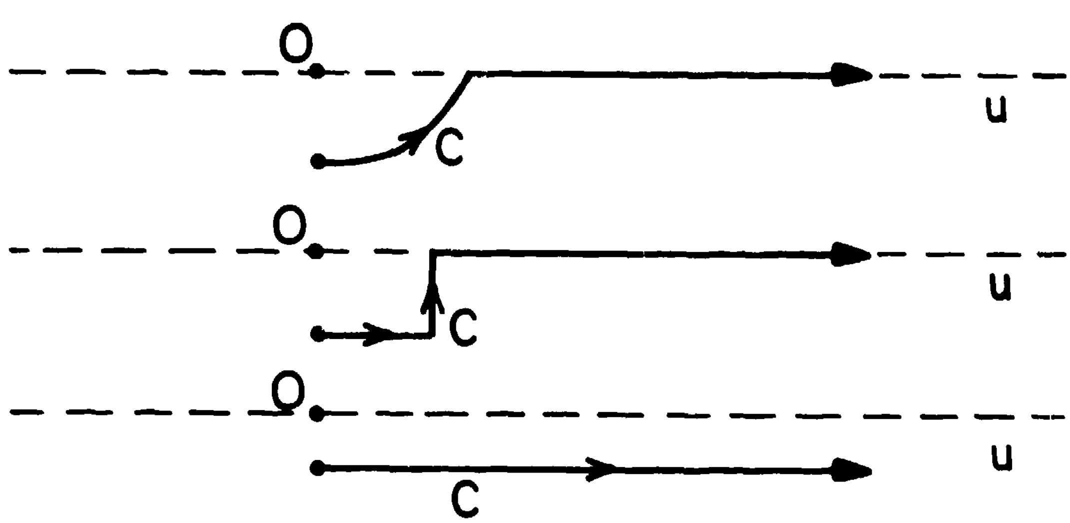

FIGURE A1

PART II

ANNOTATED BIBLIOGRAPHY OF

RESEARCH PAPERS

This part of the work contains a comprehensive, fully annotated bibliography of all research results known to the author. An exhaustive search of the free-world literature has been made. Some manuscripts which have never been published in the open literature are also included.

A few papers included deal primarily with many - versus - many situations, but contain some material on one - versus - one duels. In this case, only the latter is annotated and appears herein.

The format is as follows:

(1) Each paper has an alpha-numeric designation consisting of the first one (two if necessary) letters of all authors last names, plus a number.

The papers are arranged alphabetically by first author, starting with sole authorship and then dual authorship, and so on. The number indicates the chronological sequence of all papers by a specified author, or set of authors. The chronology is based on the date of the first citation given for each entry in the bibliography. There may be more than one citation given, as it is customary to produce first a company or university, etc., version and then an open literature publication. The latter is always the first citation. The reason all versions are given is that a reader who may have access to one version, may rest

assured that there is no difference in content between versions, except as noted.

Next, if there are both fundamental marksman (FM) and fundamental duel (FD) results, the FM part is given first and the FD portion is next. The format for each part is the same, and goes as follows:

(1) Identified as FM or FD

(2) For each model considered under FM or FD

  (a) identified as CRIFT or FIFT, or mixed

  (b) all modifications, e.g., ammunition limitations, etc.

(3) General solutions which are given

(4) Particular examples with details

Finally, at the bottom left-hand side, a listing of all the principal techniques of derivation used is given.

The fundamental marksman (FM) problem is defined as a marksman firing at a passive target. He fires at certain intervals and either hits or misses on each round. The trials cease on the first hit. The hit probability is constant from round to round. The time between rounds may be a continuous RV or a constant. He starts with an unloaded weapon, has unlimited time to hit the target and has unlimited ammunition.

The fundamental duel (FD) pits 2 marksmen, as defined above, against each other. The duel terminates when one hits the other, or both are hit simultaneously. They start at the same time and, in general, have different kill probabilities and different interfiring times.

NOTATION AND ABBREVIATIONS

CRIFT - continuous random interfiring times

cf - characteristic function $\triangleq \int_{-\infty}^{+\infty} e^{iut} f(t)dt$

CRV - continuous random variable

df - distribution function

DDC - defense documentation center (documents may be ordered from this center using numbers as shown in bibliography)

Erlang (k) - a RV with pdf given by

$$f_X(x) = \frac{\alpha^k x^{k-1}}{(k-1)!} e^{-\alpha x}, \qquad x > 0, k = 1,2,\ldots$$

$$= 0 \qquad\qquad , \qquad \text{elsewhere}$$

E[X] - expectation of X

FD	-	fundamental duel
FIFT	-	fixed interfiring times
FM	-	fundamental marksman
gt	-	geometric transform $\triangleq \Sigma f(n)z^n$, also sometimes called the z transform and if $f(n)$ are elements of a pmf, sometimes called a probability generating function
$h(t)$	-	pdf of RV, marksman's time to a kill
$H(t)$	-	df of RV, marksman's time to a kill
H	-	the event of a hit
$\overline{H}$	-	the event of no hit
IFT	-	interfiring times
iid	-	independent, identically distributed
L T	-	Laplace transform $\triangleq \int_0^\infty e^{-st} f(t)dt$
mgf	-	moment generating function (same as the Laplace transform with s replaced by -s)
MLE	-	maximum likelihood estimate
ned	-	negative exponential pdf, given by

$$f_X(x) = r e^{-rx}, \quad x > 0$$
$$= 0 \quad , \quad \text{elsewhere}$$

N	-	number of rounds fired; may or nay not be a RV (sometimes used as a general constant, as noted in text)
$N(\mu, \sigma^2)$	-	normally distributed RV with mean μ and variance σ^2
pdf	-	probability density function, denoted by $f_X(x)$
pmf	-	probability mass function, denoted by $p_X(x)$
pgf	-	probability generating function (or, sometimes called geometric transform or z transform)
P[A]	-	probability A wins the duel

P[B]	- probability B wins the duel
P[AB]	- probability neither A nor B wins the duel, consequently, a draw
r	- rate of fire or value of RV, R
R	- RV, number of hits to a kill (for a situation where more than one hit is required)
RV	- random variable
T	- RV, marksman's time to a hit (kill) - may be subscripted as appropriate
TOF	- time-of-flight (may be a RV)
V[X]	- variance of X
X	- interfiring time RV

A1. ANCKER, C.J., Jr., "Stochastic Duels With Limited Ammunition Supply," Operations Research, Vol. 12, No. 1, Jan.-Feb., 1964, pp. 38-50.

(Also, System Development Corporation, Santa Monica, CA., Document SP-1017/001/00, 23 April, 1963, 24 pp., DDC No. AD-404 104.)

FD - CRIFT - (1) Random ammunition supply

- (2) Fixed ammunition supply

- (3) Special case; same as (1) above, except duel terminates when either side runs out of ammunition before a kill

General Solutions: P[A], P[B], P[AB] (both sides run out of ammunition, (1) and (2); either side runs out of ammunition, (3))

Examples: (1) and (2) only

Distribution of Number of Rounds

A	B	IFT pdf's
Geometric	Geometric	ned
Poisson	Geometric	ned
Binomial	Geometric	ned
Geometric	Geometric	Erlang (2)
Fixed	Fixed	ned

Curves comparing this duel with FD

mixture technique
characteristic functions

A2. ANCKER, C.J., Jr., "Stochastic Duels of Limited Time-Duration," CORS Journal (Canada), Vol. 4, No. 2, July, 1966, pp. 69-81.

(Also, System Development Corporation, Santa Monica, CA., Document SP-1017/005/00, 30 March, 1964, 24 pp., DDC No. AD-436 529.)

Length of Duel

FD - (1) CRIFT (a) Continuous random time limitation
(b) Fixed time limitation

- (2) FIFT (a) Continuous random time limitation
(b) Fixed time limitation

A2. (cont'd)

General Solutions: P[A], P[B], P[AB] (both sides run out of time or kill simultaneously)

Examples:

A's IFT	B's IFT	Distribution of Time Limitation
ned	ned	ned
Erlang (2)	Erlang (2)	ned
ned	ned	fixed
Erlang (2)	Erlang (2)	fixed
fixed	fixed	ned
fixed	fixed	fixed

Curves comparing this duel with FD

mixture technique
characteristic functions
number theory

A3. ANCKER, C.J. Jr., "Stochastic Duels With Time-of-Flight Included," OPSEARCH (India), Vol. 3, No. 2, 1966, pp. 71-92.

Errata OPSEARCH (India), Vol. 3, No. 3, 1966, p. 155.

(Also, System Development Corporation, Santa Monica, CA., Document SP-1017/009/00, 19 May, 1966, 28 pp.)

3 Procedures Considered:

(1) No Delay - firing proceeds as rapidly as possible, no delay to observe effect

(2) Delay - each round is allowed to hit before next round is prepared and fired

(3) Mixed - one side delays, the other has no delay

FM-CRIFT - (1) No Delay - random TOF
- fixed TOF

- (2) Delay - random TOF
- fixed TOF

FM-FIFT - No Delay - fixed TOF

General Solutions: (pdf time to fire killing round, pdf time to kill)

A3. (cont'd)

Examples:

Procedure	IFT pdf	TOF pdf
No Delay	ned	ned
No Delay	ned	constant (fixed)
Delay	ned	ned

FD - CRIFT - (1) No Delay - random TOF
- fixed TOF

- (2) Delay - random TOF

- (3) Mixed - A fixed TOF (delay), B zero TOF
- Both fixed, A delay, B no delay

FD - FIFT - (1) No Delay - fixed TOF

- (2) Delay - fixed TOF

General Solution: P[A], P[B], P[AB] (both killed)

Examples:

Procedure	A's IFT pdf	A's TOF pdf	B's IFT pdf	B's TOF pdf
No Delay	ned	ned	ned	ned
No Delay	ned	fixed	ned	fixed
No Delay	fixed (c × B's IFT)	fixed	fixed	fixed
Delay	ned	ned	ned	ned
A Delay	ned	fixed	ned	zero
A Delay B no Delay	ned	fixed	ned	fixed

Special Case:

FD - CRIFT - no delay - TOF varies linearly

General Solution: P[A], P[B] P[AB]

Example:

A's IFT pdf	B's IFT pdf
ned	ned

mixture technique
characteristic functions
number theory

A4. ANCKER, C.J., Jr., "Stochastic Duels With Round-Dependent Hit Probabilities," Naval Research Logistics Quarterly, Vol. 22, No. 3, Sept., 1975, pp. 575-583.

(Also, University of Southern California, Los Angeles, CA., ISE Department Technical Report TR 74-3, 2 August, 1974, 16 pp.)

FM - CRIFT - Round dependent hit probabilities

General Solution: pdf time to a kill

FD - CRIFT - Round dependent hit probabilities

General Solution: P[A] P[B]

Examples:

A's IFT pdf	A's Hit Probability on j - th Round	B's IFT pdf	B's Hit Probability
ned	$q_j = \left(\frac{N}{j} - 1\right)\frac{q_A}{(N-1)}$ (N, a fixed integer)	ned	fixed
Erlang (2)	$q_j = \frac{q_A}{j}$	ned	fixed
ned	$q_j = \frac{q_A}{j}$	ned	fixed

Curves for last example and comparing 1-st with FD

mixture technique
characteristic functions

A5. ANCKER, C.J., Jr., "Stochastic Duels With Bursts," Naval Research Logistics Quarterly, Vol. 23, No. 4, Dec., 1976, pp. 703-711.

(Also, University of Southern California, Los Angeles, CA., ISE Department Technical Report TR-73-5, Nov., 1973, 14 pp.)

A5. (cont'd)

FD - Mixed, CRIFT - FIFT

General Solutions: P[A], P[B]

Example: A - FIFT, B - ned IFT

Curves of solution

FM - (1) Bursts of fixed size N (random time between rounds and random times between bursts)

- (2) Same as (1) with fixed times between rounds in a burst

General Solutions: pdf time to a kill

FD - A fires burst of fixed size N. Rounds within a burst equally spaced. Random time between bursts. B is CRIFT, no bursts.

General Solutions: P[A], P[B]

Example:

A	B IFT
ned between bursts	ned

Solution curves, comparison with FD

mixture technique
characteristic functions

A6. ANCKER, C.J., Jr., "Theory of Stochastic Duels - Miscellaneous Results," TRASANA TECHNICAL MEMORANDUM 2-77, March, 1978, 39 pp., U.S. Army TRADOC Systems Analysis Activity, White Sands Missile Range, New Mexico, DDC No. AO-52158.

FM - Erlang (n) CRIFT

General Solution: pdf and df time to a kill

Example: n = 1, n = 2, solution curves

FD - Tactical equity (each side fires first 1/2 the time and then FD starts)

General Solution: P[A]

Example: Erlang (2) CRIFT's

A6. (cont'd)

FD-CRIFT's - tactical equity (except 2-nd firer returns fire immediately)

General Solution: P[A]

Example: ned IFT's

FD - CRIFT - initial surprise a CRV

Example: IFT's - ned, surprise pdf - Laplace
- solution curves, P[A]

Sub-Example:

Surprise: (1) pdf - ned

(2) pdf - ned for negative time only

FD - Erlang (n , m) CRIFT's

General Solution: P[A], P[B]

6 problems by Thompson in T1 are simplified by using characteristic functions, viz:

(1) (a) (b) (c) and (2) (a),(b),(c)

1 problem by Thompson in T2 is simplified by using geometric transforms and characteristic functions, viz: 2 (c)

Some useful results in the Theory of Characteristic Functions are listed or derived as follows:

3 Parseval Theorems
7 properties of cf's of positive RV's
15 theorems concerning cf's of positive RV's
2 theorems on contour integration in the complex plane for special integrands are developed. Useful in numerical integration.

mixture technique
characteristic functions
geometric transform

A & Gl. ANCKER, C.J., Jr., and GAFARIAN, A.V., "The Distribution of Rounds Fired in Stochastic Duels," Naval Research Logistics Quarterly, Vol. 11, No. 4, Dec., 1964, pp. 303-327.

(Also, Systems Development Corporation, Santa Monica, CA., Document SP-1017/004/00, 4 March, 1964, 35pp., DDC No. AD-433 764.)

FM - CRIFT)- Random ammunition supply (contains fixed
- FIFT) supply case)

General Solutions: $P(H)$, $P(\overline{H})$, $P(N = n \mid H)$

$E[N \mid H]$, $E[N^2 \mid H]$,

$P[N \geq n_0 \mid H]$, $P[N = n \mid \overline{H}]$, $P[N = n]$

Examples: Distribution of N (no. of rounds fired)

(1) Geometric
(2) Infinite supply
(3) Finite, fixed supply

FD - (1) CRIFT - (a) random ammunition supply
(b) fixed ammunition supply

General Solutions: $P[N_A = n \mid A]$, $P[N_A \geq n_0 \mid A]$, $P[N_A = n \mid AB]$,

$P[N_A = n \mid B]$, $P[N_A = n]$

$E[N_A \mid A]$, $E[N_A^2 \mid A]$

Marginal increase in P[A] if ammunition supply is increased.

Examples:

	A		B	
	pmf of N_A	IFT pdf	pmf of N_B	IFT pdf
(1)	Geometric	ned	Geometric	ned
(2)	Binomial	ned	Infinite (no limitation)	ned
(3)	Fixed (constant)	ned	Fixed (constant)	ned

FD - (2) FIFT - (a) random ammunition supply
(b) fixed ammunition supply

A & G1. (cont'd)

General Solutions: $P[N_A = n \mid A]$, $P[A]$, $P[AB]$, $P[N_A = n \mid AB]$,
$P[N_A = n \mid B]$, $P[N_A = n]$

Marginal increase in $P[A]$ if ammunition supply is increased.

Examples:

	A pmf of N_A	B pmf of N_B
(1)	Geometric	Geometric
(2)	Infinite supply	Infinite supply
(3)	Fixed (constant)	Fixed (constant)

mixture technique
characteristic functions
number theory

A & G2. ANCKER, C.J., Jr., and GAFARIAN, A.V., "The Distribution of the Time-Duration of Stochastic Duels," Naval Research Logistics Quarterly, Vol. 12, Nos. 3 & 4, Sept.-Dec., 1965, pp. 275-294.

(Also, System Development Corporation, Santa Monica, CA., Document SP-1017/007/00, 10 August, 1964, 29 pp., DDC No. AD-606 169.)

FM - CRIFT } (1) random time limitation
 - FIFT } (2) fixed time limitation

General Solutions: $P[H]$, pdf time to hit, pdf time for no hits, all moments of two preceding pdf's (special case, no time limit), pdf of total time to completion

Example:

IFT pdf	pdf - time limit
Erlang (2)	ned
Fixed	ned

A & G2. (cont'd)

FD - (1) CRIFT (a) random time limitation
(b) fixed time limitation

- (2) FIFT (a) random time limitation
(b) fixed time limitation

General Solutions: pdf of $T_A | A$, pdf of $T_{AB} | AB$, all moments of the preceding pdf's (special case, no time limit), total time pdf.

Example:

	A's IFT pdf	B's IFT pdf	pdf - time limit
(1)	ned	ned	ned
(2)	Erlang (2)	Erlang (2)	ned
(3)	ned	ned	fixed
(4)	Erlang (2)	Erlang (2)	fixed
(5)	fixed (integer × B's IFT)	fixed	ned
(6)	fixed (integer × B's IFT)	fixed	fixed
(7)	fixed	fixed	none (infinite time)

mixture technique
characteristic functions
number theory

A & W1. ANCKER, C.J., Jr., and WILLIAMS, Trevor, "Some Discrete Processes in the Theory of Stochastic Duels," Operations Research, Vol. 13, No. 2, Mar.-Apr., 1965, pp. 202-216.

(Also, System Development Corporation, Santa Monica, CA., Document SP-1017/002/00, 13 August, 1963, 28 pp., DDC No. AD-420 514)

FD - (1) FIFT

General Solutions: P[A], P[B], P[AB]

Examples:

(1) A's IFT an integral multiple of B's IFT

(2) B's IFT an integral multiple of A's IFT

Solution curves

FD - (2) Equal FIFT - probability of a near miss is included. A near miss causes a displacement and the loss of one firing turn

General Solutions: P[A], P[B], P[AB]

Solution curves

mixture procedure
number theory
stochastic difference equations for (2)

Ba 1. BARFOOT, C.B., "The Lanchester Attrition Rate Coefficient: Some Comments on Seth Bonder's Paper and a Suggested Alternate Method," Operations Research, Vol. 17, No. 5, Sept.-Oct., 1969, pp. 888-894.

FM - Markov dependent states > 2 - IFT constant but dependent on current state

Kill probabilities dependent on current state
Initial conditions may be varied

General Solution: E[T]: in matrix form

Example: Numerical

Markov chain theory
matrix algebra

Ba 2. BARFOOT, C.B., "Markov Duels," Operations Research, Vol. 22, No. 2, Mar.-Apr., 1974, pp. 318-330.

(Also, "Stochastic Duels in Which Each Contestant's Shots Form a Markov Chain," OR-69, 5-th International Conference on Operations Research, Venice, Italy, 23-27 June, 1969, ed. by John Lawrence, Tavistock Publ., London, 1970, pp. 223-234.)

(Also, Master's Degree Thesis submitted to the Department of Operations Research, George Washington University, Washington, D.C.)

(Also, "Stochastic Duels With Markov Dependent Kill Probabilities," Center for Naval Analyses Working Paper, Arlington, VA., undated, 48 pp.)

FM - FIFT - Markov dependent states > 2

General Solutions: pmf (number of rounds to a kill or time to kill); in matrix form E(N), V(N)

FD - FIFT - Markov dependent states > 2

General Solutions: P[A], P[B], P[AB] in matrix form

Example: Two-state case FM-FIFT result

FD - FIFT - Markov dependent states > 2. A fires y rounds first (random initial surprise), pmf(Y) - geometric

General Solutions: P[A], P[B], P[AB] in matrix form

Numerical example

Ba 2. (cont'd)

Markov chain theory
matrix algebra

Ba 3. BARFOOT, C.B., "Some Anti-Armor Models Used in U. S. Marine Corps Planning Studies," MCOAG CNA, Arlington, VA., NATO Conference, 26-30 August, 1974, Munich, Germany, 19 pp.

FD - FIFT - Markov dependent states > 2

A fires busts of constant length, with a constant time between rounds and a constant time between bursts

B fires with no burst, just constant time between rounds

General Solutions: P[A], P[B], P[AB] in matrix form; asymptotic approximations to general solutions

Same as above - but A fires y rounds first (surprise)

(a) y a constant

(b) Y a geometric random variable

General Solutions: P[A], P[B], P[AB] in matrix form; asymptotic approximations to general solution

FM - CRIFT - Markov dependent states > 2

IFT's are ned and dependent on current state (different parameter for each state)

General Solution: h(t)

FD - CRIFT - Markov dependent states > 2

Each duelist's IFT's are ned and dependent on current state (different parameters for each state)

General Solution: P[A], P[B]; matrix double integral and a closed solution by similarity transformation

FD - CRIFT - Markov dependent states > 2

Each duelist's IFT's are ned and state dependent:

(1) A has a fixed time to fire first (surprise)
(2) A has a random surprise time with ned distribution

Ba 3. (cont'd)

General Solution: P[A], P[B]; matrix double integral and closed form by similarity transformation

Markov chain theory
semi-Markov chain theory
matrix algebra

Bh 1. BHASHYAM, N., "Stochastic Duel With Several Types of Weapons," Defence Science Journal (India), Vol. 17, No. 2, April, 1967, pp. 113-118.

(Also, Defence Science Laboratory Report, Delhi-6, India, 9 pp.)

(Also, Bh 3, pp. 67-72.)

FM - CRIFT - several weapons firing simultaneously and independently, each with individual ned IFT's and kill probabilities.

General Solution: h(t)

FD - CRIFT - each side with several (different for each side) weapons, firing simultaneously and independently. Each weapon with individual ned IFT's and kill probabilities.

General Solution: P[A], P[B]

Mean and variance of number of rounds of each type weapon fired to a kill.

differential difference equation technique
z transforms

Bh 3 uses elementary methods (much simpler but cannot get last results above.

Bh 2. BHASHYAM, N., "Stochastic Duels With Pattern Firing," Advancing Frontiers in Operational Research, Proceedings of the International Seminar on Operational Research, New Delhi, India, 7-10 August, 1967, Ed. by H.S. Subba Rao, N.K. Jaiswal, and A. Ghosal, Hindustani Publishing Corporation (India), 1969, pp. 151-164.

(Also, Bh 3, pp. 100-114)

FM - CRIFT - two weapons fired alternately, each firing a fixed number of rounds (different for each weapon) with a different IFT pdf and a different hit probability.

General Solution: LT h(t)

FD - CRIFT - each side, two weapons fired alternately, each firing a fixed number of rounds (different for each weapon) with a different IFT pdf and a different hit probability

General Solution: LT time to a kill by A
P[A], P[B]

Bh 2. (cont'd)

Examples:

(a) Each side fires one round each with two weapons and ned IFT's
(b) A has one weapon, one round each
B has two weapons, one round each
(c) A has two weapons, one round each
B has one weapon, one round each

Numerical example, curves

Bh 3 - different numerical examples and curves

supplementary variable technique
differential difference equations
geometric transforms
Laplace transforms

Bh 3. BHASHYAM, N., "Stochastic Duels," Ph.D. Thesis, University of Delhi, Delhi, India, May, 1969, 186 pp.

Four sections in this thesis have not been published in the open literature. Only these are annotated here. The other sections are cross referenced in the appropriate document.

(1) "Stochastic Duels With Only Pooled Interfiring Time Distribution Known," pp. 73-84.

FD - CRIFT - Probability that either fires next, given either fired last, is probablistic. Therefore, the sequence of firings is Markov and independent of times between rounds. IFT's are different, depending on who fired last.

General Solution: LT (1) time to win by A (i.e., during duel)
(2) time to win by B (i.e., during duel)

Numerical example - curves

differential difference equations
supplementary variables
geometric transform
Laplace transforms

(2) "Stochastic Duels With Burst Fire," pp. 87-100

Bh 3. (cont'd)

FM - CRIFT - Bursts of fixed length fired with CRIFT and with continuous RV between bursts

General Solution: LT h(t)

FD - CRIFT - Bursts of fixed length fired with CRIFT and with continuous RV between bursts

General Solution: LT P[A], P[B]

Examples: (1) all times ned, burst length very large
(2) all times ned, A large burst size, B single round fire (no bursts)

Numerical results - curves

differential difference equations
supplementary variable technique
geometric transform
Laplace transforms

(3) "Stochastic Duels of Limited Time Duration and Finite Ammunition Supply," pp. 141-153.

FD - CRIFT (ned both sides) - Both fixed ammunition limitation - time limitation a continuous RV with ned pdf. A draw occurs if time runs out or both run out of ammunition

General Solution: P[A], P[B], P[AB]

Examples: (1) unlimited time
(2) B unlimited ammunition - solution curves
(3) B unlimited ammunition - unlimited time
(4) both unlimited ammunition

differential difference equations
Laplace transforms

(4) "Stochastic Duels with Repairable Weapons," pp. 153-169.

FM - CRIFT - (a) Fixed limited ammunition
(b) ammunition limitation a discrete RV

Time to failure of weapon pdf is ned. Repair time is CRV

General Solution: LT h(t) and time to run out of ammunition

FD - CRIFT - (a) Fixed limited ammunition (both)
(b) Ammunition limitation a discrete RV

Bh 3. (cont'd)

Time to failure of weapons are pdf's with ned's. Repair times are CRV's. Contestant under fire during repair time.

General Solutions: P[A], P[B], P[AB]

Particular Cases:

FM & FD - (1) A unlimited ammunition; B fixed ammunition limit
FM & FD - (2) Both unlimited ammunition

Examples: (1) Unlimited ammunition, IFT's ned
Repair times ned
(2) Unlimited ammunition, IFT's ned
A repair time ned; B failure free

Solution curves

differential difference equations
supplementary variable technique
geometric transforms
Laplace transforms

Bh 4. BHASHYAM, N., "Stochastic Duels With Single Shot Kill Probability Varying As A Function of Inter-Firing Time Interval," Defence Science Laboratory, Delhi-6, India. Draft - Private Communication, Spring 1970, 10 pp.

FM - CRIFT - Fixed ammunition limitation
- Kill probability a function of IFT

General Solution: LT h(t)

FD - CRIFT - Fixed ammunition supplies, both sides
- Kill probabilities, functions of IFT's

General Solutions: Integrals of LT P[A], P[B], P[AB]

Example: Infinite ammunition supplies
IFT's are ned
Kill probabilities a negative exponential function of IFT's

differential difference equations
geometric transforms
Laplace transforms
supplementary variable technique

Bh 5. BHASHYAM, N., "Stochastic Duels With Round Dependent Kill Probability and General Inter-Firing Times," Defence Science Laboratory, Delhi-6, India. Draft - Private Communication, Spring 1970, 15 pp.

FM-CRIFT - Ammunition limitation: (a) fixed, and (b) random
Round-dependent hit probabilities

<u>General Solutions</u>: LT h(t)

FD-CRIFT - Ammunition limitation: (a) fixed, and (b) random
Round-dependent hit probabilities

<u>General Solutions</u>: P[A], P[B], P[AB]; also,

(1) A fixed ammunition limit
B infinite supply
(2) Both have infinite supply

<u>Examples</u>: (1) both fixed ammunition supply, ned IFT's (solution in terms of unspecified hit probabilities)

(2) both fixed ammunition supply and both general Erlang IFT's (solution in terms of unspecified hit probabilities)

differential difference equations
special discrete transforms
geometric transforms
Laplace transforms
supplementary variable technique

Bh 6. BHASHYAM, N., "Stochastic Duels With Non-Repairable Weapons," <u>Naval Research Logistics Quarterly</u>, Vol. 17, No. 1, March, 1970, pp. 121-129.

(Also, Defence Science Laboratory Report, Delhi-6, India, undated, 13 pp.)

(Also, Bh 3, pp. 169-181.)

FM-CRIFT (ned only) - Limited ammunition, failure prone weapons with limited replacement stock (failure times are ned)

<u>General Solutions</u>: LT h(t)
LT time-to-failure (weapons or ammunition supply)

Bh 6. (cont'd)

FD - CRIFT (ned only) - Limited ammunition, failure prone weapons with limited replacement stock (failure times are ned)

General Solutions: P[A], P[B], P[AB]

Also, same with unlimited ammunition and replacement stock a discrete RV

Examples: (1) both unlimited ammunition, A limited fixed weapon supply

numerical illustration with curves

(2) from Bh 3 - both unlimited ammunition, both geometrically distributed number of weapons

differential difference technique
Laplace transforms

Bh 7. BHASHYAM, N., "Stochastic Duels With Lethal Dose," Naval Research Logistics Quarterly, Vol. 17, No. 3, Sept., 1970, pp. 397-405.

(Also, Bh 3, pp. 114-125.)

FM - CRIFT - Multiple hits to a kill (R)

(a) fixed R
(b) R a discrete RV

General Solution: LT h(t)

FD - CRIFT - Multiple hits to a kill (R)

(a) fixed R
(b) R a discrete RV

General Solutions: P[A], P[B]

Examples: (1) IFT ned for both, R fixed for both
Solution curves

(2) IFT ned for both, R a geometric RV for both
Solution curves

geometric transforms
differential difference equations
Laplace transforms

Bh 3 gives a much simpler derivation

Bh 8. BHASHYAM, N., "Stochastic Duels With Correlated Fire," Metrika, Vol. 20, No. 1, February, 1973, pp. 17-24.

(Also, Bh 3, pp. 125-137.)

FM - CRIFT - Two weapons with different IFT's and different hit probabilities. The probability of firing a given weapon on the next round, given a particular weapon was fired on the last round, is fixed. This leads to a correlation between sequences of weapons fired.

General Solution: LT h(t)

FD - CRIFT - Two weapons with different IFT's and different hit probabilities. The probability of firing a given weapon on the next round, given a particular weapon was fired on the last round, is fixed. This leads to a correlation between sequences of weapons fired.

General Solution: P[A], P[B]

Examples: ned IFT's for both

differential difference equations
geometric transforms
Laplace transforms
supplementary variables

Bh & Si 1. BHASHYAM, N., and SINGH, N., "Stochastic Duels With Varying Single Shot Kill Probabilities," Operations Research, Vol. 15, No. 2, Mar.-April, 1967, pp. 233-244.

(Also, Defence Science Laboratory Report, Delhi-6, India, November, 1966, 19 pp.)

FD - CRIFT (ned only) - Fixed ammunition limitation

- Kill probability is a function of round number

General Solution: (1) P[A], P[B], P[AB]

(2) B has infinite ammunition supply P[A], P[B]

(3) A and B have infinite ammunition supplies; P[A], P[B] developed separately without LT

Examples: (1) fixed kill probabilites; B infinite ammunition supply; also, both infinite ammunition supply

(2) $p(n) = \frac{1}{n+1}$; $p'(m) = \frac{1}{m+1}$

(3) $p(n) = (1 - \alpha^n)$; $p'(m) = (1 - \beta^m)$

α, β parameters

differential difference equations
special discrete transforms
Laplace transforms

Bo 1. BONDER, Seth, "The Lanchester Attrition-Rate Coefficient," Operations Research, Vol. 15, No. 2, March-April, 1967, pp. 221-232.

FM - FIFT - Markov dependent hit probabilities (dependent on states) three states

Fixed number of multiple hits required to kill

General Solution: pmf of N - number of rounds fired to give a fixed number of hits

E[N]

combinatorial arguments

Bo 2. BONDER, Seth, "The Mean Lanchester Attrition Rate," Operations Research, Vol. 18, No. 1, Jan.-Feb., 1970, pp. 179-181.

General Solution: If T is RV, time to a kill in situation of Bo 1, gives E[T]

elementary probability arguments

Fi 1. FINLEY, David R., "A Theoretical Study of Round-to-Round Correlation in Gunnery," Cornell Aeronautical Laboratory, Inc., Buffalo, New York, Internal Research Report, WA-86-184, Nov., 1968, 33 pp.

(Also, Master of Arts Thesis at the American University, 10 May, 1968, 33 pp. Available at University Microfilms, Inc., Ann Arbor, Michigan, No. M1647)

FM - CRIFT

3 states { miss; hit, not killed; hit, killed }

P[hit] = p, a constant; P[miss] = 1 - p; and P[kill | hit] = θ, a constant. However, p may depend on any or all of the previous rounds fired. General results are derived which do not apply directly to FM or FD because more than one killing round is allowed. However, these results can be adapted to FM or FD. Very generally, positive correlation is defined as:

P[hit on i-th round | hits on specified previous rounds]

≥ P[hit on i-th round | miss on at least one of the specified previous rounds and miss on all other rounds].

Note: (a) any hit <u>may</u> be a kill
(b) this is not the same as the usual definition of correlation and the results do not apply to ordinary correlation.

<u>General Solution</u>: Positive correlation decreases kill probability, compared to the case where all trials are independent. For usual definition of correlation, positive correlation does not necessarily decrease kill probability.

<u>Example</u>: no overkills (i.e., first kill terminates process) dependence is Markov

<u>Results</u>: positively correlated if correlation coefficient ≥ 0; distribution of N, E[N], V[N]

<u>Example</u>: cf of h(t) for general IFT and E[T], V[T]; also ned

FD - A is FM as above, B is FM
Both IFT's ned

General Solution: P[A]

set theory

Fr 1. FRIEDMAN, Yoram, "A Model for the Determination of Optimal Inter-firing Times," (unpublished ms.), Faculty of Management, Tel Aviv University, Tel Aviv, Israel, July, 1976, 8 pp.

FM - Fixed time limit
- Kill probability a monotone non-decreasing function of IFT
- IFT is continuous but greater than a certain minimum and at choice of firer

For a fixed number of rounds fired (n) [$\leq$ the maximum possible in time limit] proves that optimal kill policy is to continually decrease the IFT's, but use up all the time available.

Solution: Gives an algorithm for finding the t's. Also shows how to find the best n.

Lagrange multipliers

Ga & Sl. GARG, R.C., and SINGH, N., "A Stochastic Duel in a Hunter-Killer Game - I," The Symposium on Operations Research, No. 42, 1970, pp. 183-192

FD - CRIFT (both ned)

3 states
(1) combat - time is ned
(2) no contact - time is ned
(3) reclose & continue time is general pdf
all CRV's

General Solutions: LT's of time-functions of various states; in particular that A or B has won at time t after each has expended a certain number of rounds

Examples: (1) reclose and continue time is ned (complete solution)

(2) same as (1), but FM only

differential difference equations
Laplace transforms
supplementary variable technique
geometric transforms

Gr 1. GROVES, Arthur D., "The Mathematical Analysis of a Simple Duel," Ballistic Research Laboratory Report No. 1261, Aberdeen Proving Grounds, Md., August, 1964, 23 pp., DDC No. AD-609 195.

FD - FIFT - A starts with a fixed time advantage (time he fires before B does)

General Solution: in matrix form

(1) state probabilities after m cycles

(2) P[A], P[B], P[AB]

Numerical examples

Markov chains
number theory - matrix manipulations
geometric transforms

H 1. HARRIS, Terrell J., "Many Versus Many Stochastic Duels," Caywood-Schiller Associates Report, Chicago, Illinois, Fall 1967, 18 pp.

FD - CRIFT - A general IFT

B ned IFT

<u>General Solutions</u>: for P[A], P[B] in LT and geometric transforms for mean number of rounds fired to a victory for A

<u>Examples</u>: (1) A's IFT - Erlang (k)

(2) A's IFT - mixture of ned's

renewal theory
geometric transforms
Laplace transforms

J & Bl. JAISWAL, N.K., and BHASHYAM, N., "Stochastic Duels With Flight Time and Replenishment," OPSEARCH (India), Vol. 3, No. 4, 1966, pp. 169-185.

(Also, Defence Science Laboratory Report, Delhi-6, India, 1966, 22 pp.)

(Also, BH 3, pp. 45-67)

FM - CRIFT - fixed ammunition supply at start

- ammunition replenishment of a fixed amount at times with pdf (ned)

- TOF is CRV

General Solution: LT of h(t)

Special cases:

(1) no replenishment, LT of h(t)

(2) no replenishment, initial supply a discrete RV (in BH 3 only), LT of h(t)

(3) no replenishment, unlimited ammunition, LT of h(t)

(4) zero flight-time, LT of h(t)

(5) zero flight-time, no replenishment and initial supply is fixed, LT of h(t)

(6) zero flight-time, no replenishment and initial supply is a discrete RV, LT of h(t)

(7) flight-time zero, no replenishment, unlimited ammunition, LT of h(t)

FD - CRIFT - fixed ammunition supply at start

- ammunition replenishment of a fixed amount at times with pdf (ned)

- TOF is CRV

General Solutions: P[A], P[B], P[AB]

Examples:

(1) $\begin{cases} \text{IFT's ned} \\ \text{TOF's ned} \end{cases}$ - unlimited ammunition

(2) $\begin{cases} \text{IFT's ned} \\ \text{A's TOF ned} \end{cases}$ - unlimited ammunition, zero flight-time for B

curves in BH3

(3) IFT's ned - both zero flight-time, unlimited ammunition

(4) IFT's ned - both flight-times zero (in Bh3)
- B unlimited ammunition

differential difference equations
geometric transforms
Laplace transforms
supplementary variables

Ki 1. KIMBLETON, Stephen R., "Attrition Rates for Weapons With Markov-Dependent Fire," Operations Research, Vol. 19, No. 3, May-June, 1971, pp. 698-706.

FM - FIFT - Markov dependent hit probabilities (dependent on states) 3 states, random number of multiple hits to kill

General Solutions: Laplace transform of N (number of rounds to a kill), E[N], V[N], pmf of N

If T is time-to-kill, Laplace transform of pmf of T, E[T], V[T]

Markov chain theory
renewal theory
difference equations
Laplace transforms
geometric transforms

KW & B1. KWON, T.Y., and BAI, D.S., "Stochastic Duels With Multiple Hits and Limited Ammunition Supply," Korea Advanced Institute of Science Report, Seoul, Korea, April, 1978, 23 pp.

FM - CRIFT (1) Fixed number of hits to a kill

(2) Random number of hits to a kill - geometrically distributed, i.e.: Parameter = Pr[kill | hit]

(3) Fixed number of hits to a kill; ammunition limitation a discrete RV

(4) Random number of hits to a kill (geometrically distributed); ammunition limitation (geometrically distributed)

(5) Fixed number of rounds fired simultaneously (pattern); fixed probability of a pattern hitting, each round in pattern which hits has fixed probability of a kill

(6) Limited ammunition a discrete RV; fixed number of rounds fired simultaneously with fixed probability of a pattern hitting and each round in a pattern kills with a fixed probability

General Solutions: LT h(t)

FD - CRIFT - Both sides same as FM (1), (2) and (3)

General Solutions: P[A], P[B], P[AB]

Examples: (a) fixed number of hits to a kill; ammunition limit a geometric discrete RV; IFT's ned

(b) geometric number of hits to a kill; geometric distribution for number of rounds, IFT's ned - curves

(c) pattern firing (5) above, with IFT's ned

(d) pattern firing (6) above, with geometric distribution for number of rounds, IFT's ned - curves

mixture technique
Laplace transforms
Case (2) also derived using renewal theory

M&A1. MOHAN, C., and ARORA, S.D., "On a Problem in Naval Defense," Operations Research, Vol. 12, No. 2, March-April, 1963, pp. 194-198.

FM-CRIFT (ned)

2 states { (1) combat - time length is ned ; (2) between engagements - time length general pdf } CRV's

Although IFT is ned, only firing times when hits occur are considered, so these events are ned with a parameter pr (not r)

General Solution: LT of time to n hits

Example: between - between-engagements time ned
also, expected time for n hits

differential difference equations
supplementary variable technique
Laplace transforms
geometric transforms

N & J1. NAGABHUSHANAM, A., and JAIN, G.C., "Stochastic Duels With Damage," Operations Research, Vol. 20, No. 2, March-April, 1972, pp. 350-356.

(Also, Defence Science Laboratory Report, Delhi-6, India, undated, 13 pp.)

FM-CRIFT (1) Amount of damage per round is a discrete RV with a pmf. Damage is independent round-to-round and cumulative until total is a kill (predetermined total damage)

(2) Damage per found is a CRV with pdf; otherwise, same as (1) above

(3) Damage is time-homogeneous, i.e., damage increase in Δt = p(increase)Δt + O(Δt), where increase is a discrete RV

General Solutions: cf h(t)

Example: three discrete damage states for (1), IFT ned

FD-CRIFT - same for both as (1), (2) and (3) above, plus

(4) Damage states are: no damage, damage, Kill. Given damage, the amount is a CRV with a cumulative upper limit causing a kill.

General Solutions: P[A], P[B]

Examples: (a) three discrete damage states for (1), IFT's ned

(b) for (3) above, two discrete damage states

difference equations
geometric transforms
renewal theory
differential difference equations
characteristic functions

R & S1. RUSTAGI, J.S., and SRIVASTAVA, R.C., "Parameter Estimation in a Markov-Dependent Firing Distribution," Operations Research, Vol. 16, No. 6, Nov.-Dec., 1968, pp. 1222-1227.

FM - FIFT - Two Markov-dependent states

- Multiple hits (r) to get a kill, different first round hit probability

General Solution: pgf (geometric transform) of N (number of rounds to get r hits)

MLE of the three parameters involved

Markov chain theory
renewal theory
geometric transforms

Sa 1. SAVIR, David, "Asynchronous Dodging Duels," private communication, 19 August, 1970, pp. 54-69.

FD - FIFT - equal IFT's

A starts first; B starts later, by an amount < IFT

Each can either hit, miss or near-miss, causing a displacement and the loss of one firing time (while remaining vulnerable.)

<u>General Solutions</u>: P[A], P[B]

Boolean algebra

Sc 1. SCHODERBEK, J.J., "Some Weapon System Survival Probability Models - I. Fixed Time Between Firings," Operations Research, Vol. 10, No. 2, March-April, 1962, pp. 155-167.

FD - FIFT - A has a fixed time advantage (surprise) over B

(1) Equal firing times (a), A's advantage $< a$

General Solution: P[A survives to time t]

P[A], same for B

Numerical examples and example a, b not equal (special case where a and b are fixed IFT's)

(2) Same as (1) above, with lethal radius from a circular Normal(zero, σ^2) pdf

A evacuates after firing k rounds

General Solution:

P[A is alive time T after initiating evacuation]

Numerical examples

(3) Equal firing times; A fires first with probability p. A evacuates after $k > 3$ rounds, evacuation distance RV

General Solution:

P[A is alive time T after initiating evacuation | A is alive to evacuate]

Numerical example

combinatorial methods

Sc 2. SCHODERBEK, J.J., "Some Weapon System Survival Probability Models - II. Random Time Between Firings," Operations Research, Vol. 10, No. 2, March-April, 1962, pp. 168-179.

FD - CRIFT (both ned)

(1) Kill probabilities and rates of fire are continuous functions of time

General Solutions: P[A is alive at time t],

P[A], same for B

E[time to kill]

Sc 2. (cont'd)

<u>Numerical Examples</u>: kill probabilities and rates of fire constant

FD - CRIFT (both ned)

(2) Same as (1), but A evacuates at time t_0 (A is vulnerable during evacuation). A moves a distance which is Rayleigh distributed.

<u>General Solution</u>: P[A survivies evacuation | A is alive to start] and P[A survives evacuation]

differential equations

Sr & Gl. SRIVASTAVA, S.S., and GARG, R.C., "Two-Sided A/S Warfare With Limited Ammunition and Range of Fire Power," <u>Trabajos de Estadistica y de Investigacion Operativa</u>, Vol. XXIII, Madrid, Spain, 1972, pp. 135-147.

FD - CRIFT - IFT's (both ned) - both limited ammunition

4 states:

(1) Seeking contact - time is general pdf
(2) Closing in - time is general pdf
(3) Combat - time is ned
(4) Release from combat - time is ned (before starting (1) again)

All CRV's

<u>General Solutions</u>: LT's of time function of various states given above. In particular, that A wins or B wins, or a draw has occurred at time t, after each has fired a specified number of rounds

<u>Example</u>: seeking contact and closing in; both ned; general solutions as given above, inverse LT's of these functions and same functions at $t = \infty$

differential difference equations
supplementary variable technique
Laplace transform

Sr, G
& Sl. SRIVASTAVA, S.S., GARG, R.C., and SINGH, N., "A Stochastic Duel In a Hunter-Killer Game - III," Cahiers du Centre d'Etudes de Recherche Operationnelle (Belgium), Vol. 11, No. 2, July, 1969, pp. 104-111.

FD - CRIFT's (both ned)

4 states:

(1) Closing

(2) Combat

(3) No contact

(4) Reclose & continue

All these times are CRV's with general pdf's

General Solutions: Bivariate geometric transform of LT of time functions of various states. In particular, that A (or B) has won at time t after each has expended a specified number of rounds.

differential difference equations
supplementary variables technique
geometric transforms
Laplace transform

T1. THOMPSON, David E., "Reliability and Mobility in the Theory of Stochastic Duels," Part F, Ch. 1, pp. 573-612, in <u>Development of Analytical Models of Battalion Task Force Activities</u>, ed. by S. Bonder and R. Farrell. Report No. SRL 1957, FR 70-1(U), Systems Research Laboratory, Dept. of Industrial Engineering, University of Michigan, September, 1970, DDC No. AD714677.

(Also, "Mobility and Reliability in the Theory of Stochastic Duels," Master's Thesis, Dept. of Industrial Engineering, University of Michigan, Ann Arbor, Michigan, 8 August, 1968.)

(Also, DSL 02147 WP 68-4(U), Defense Systems Laboratory, University of Michigan, Ann Arbor, Michigan, 31 October, 1968.)

(Also, "Reliability and Mobility in the Theory of Stochastic Duels," a chapter, pp. 573-612, in <u>Development of Models for Defense Systems Planning</u>, ed. S. Bonder and R. Farrell, Technical Report (U) SRL 2147 TR 70-2, Systems Research Laboratory, Dept. of Industrial Engineering, University of Michigan, Ann Arbor, Michigan, September, 1970.)

FD - CRIFT

(1) Weapon life time a CRV

(a) no withdrawal

<u>General Solution</u>: P[A], P[B], P[AB] in terms of pdf's and df's

<u>Example</u>: IFT's - ned, weapon lifetimes ned

(b) withdrawal at failure times

<u>General Solution</u>: P[A], P[B], P[AB] in terms of pdf's and df's

<u>Example</u>: IFT's - ned, weapon lifetimes ned.
Can be shown to be same as A2(1)(a)

(c) withdrawal at next firing time after failure

<u>General Solution</u>: P[A], P[B], P[AB] in terms of pdf's and df's

<u>Example</u>: IFT's - ned, weapon lifetimes - ned.
Discussion of significance of failure rates.

T1. (cont'd)

(2) Round-dependent failures - round number on which failure occurs is a discrete RV

(a) no withdrawal

<u>General Solutions</u>: P[A], P[B], P[AB] in terms of pdf's and df's

Shown to be same as A1

(b) withdrawal at failure times (no delay); same as A1 special case

(c) withdrawal at next round after failure - fixed kill probabilities

<u>General Solutions</u>: P[A], P[B], P[AB] in terms of pdf's and df's

<u>Example</u>: IFT's ned

FM - CRIFT (ned)

(3) Time-dependent kill probabilities

<u>General Solution</u>: h(t) in terms of pdf's and df's

FD - CRIFT's (ned) - Time-dependent kill probabilities

<u>General Solutions</u>: P[A], P[B], P[AB] in terms of pdf's and df's

<u>Example</u>:

$$P_A(t) = \frac{a}{(r_s + vt^2)} \quad \text{, similarly, for } B$$

(a, r_s and v, constants)

FM - CRIFT (ned)

(4) Time-dependent kill probabilities and time-dependent firing rates

<u>General Solution</u>: h(t) in terms of pdf's and df's

T1. (cont'd)

FD - same, and special case: $\dfrac{P_A(t)\, r_A(t)}{P_B(t)\, r_B(t)} = k$

elementary probability arguments
mixture technique
(3) uses stochastic differential equation method

T2. THOMPSON, David E., "Stochastic Duels Involving Reliability," Naval Research Logistics Quarterly, Vol. 19, No. 1, March, 1972, pp. 145-148.

Covers some but not all of T1

One new case, 2(c), round-dependent withdrawal at next firing time after failure and probability of failure on any given round is a discrete RV

General Solutions: P[A], P[B], P[AB] in terms of pdf's and df's

mixture technique
elementary probability arguments

W1. WILLIAMS, Trevor, "Stochastic Duels - II," System Development Corporation, Santa Monica, CA., Document, SP-1017/003/00, 13 Sept., 1963, 61 pp. DDC No. AD420-515.

FM - CRIFT - Cumulants of h(t) (up to the fourth) derived in terms of cumulants of IFT pdf

FD - CRIFT - In times-to-kill as Erlang (k) functions

General Solution: Using FM above, can be used as an approximation to any FD using means and variances of IFT's

Illustrations

FD - CRIFT - Approximate solution in terms of means and variances of IFT's and kill probabilities (2 terms of an infinite series)

mixture theory
moment generating functions
finite calculus

W2. WILLIAMS, Trevor, "Stochastic Duels With Homing," System Development Corporation, Santa Monica, CA., Document, SP-1017/106/00, 18 May, 1965, 34 pp., DDC No. AD-617-773.

(Also, "Stochastic Duels - III," System Development Corporation, Santa Monica, CA., Document SP-1017/006/00, 22 June, 1964, 72 pp. DDC No. AD-443-754.)

FD - CRIFT (both ned)

Hit probability, (p_n), round-dependent and increasing, with probability of a hit on n-th round (Π_n) a discrete RV with pmf a negative binomial

p = hit PR $\triangleq \frac{1}{\bar{n}}$ where $\bar{n} = E(N)$, N the RV, round on which hit occurs

This causes $p_1 < p < p_\infty$

General Solution: P[A], P[B]

- Curves comparing outcome vs p_A, p_b (instead of p_{A1}, p_{A2}) for various parameter values
- MLE of $\hat{p}$ and $\hat{k}$ where k is a parameter in Erlang (k)

mixture theory
moment generating functions
geometric transforms

W&A1. WILLIAMS, Trevor, and ANCKER, C.J., Jr., "Stochastic Duels," Operations Research, Vol. 11, No. 5, Sept.-Oct., 1963, pp. 803-817.

(Also, System Development Corporation, Santa Monica, CA., Document SP-1017/000/01, 20 March, 1963, 22 pp., DDC No. AD-400 637.)

FD - CRIFT

General Solution: P[A], P[B]

Examples: (1) IFT's both ned - curves

(2) IFT's both Erlang (2) - curves

FD - CRIFT - Both fire simultaneously at time zero ("classical duel")

General Solution: P[A], P[B], P[AB]

Example: IFT's both ned - curves

FD - CRIFT - A fires first 1/2 the time, B fires first 1/2 the time (tactical equity)

General Solution: P[A], P[B]

Example: IFT's both ned - curves

FD - CRIFT - For a random initial period of time, one side or the other may fire with impunity (i.e., the other side cannot return fire) - called "Random initial surprise" - time advantage a CRV - positive values are A's advantage, negative values are B's advantage

General Solution: P[A], P[B]

Example: IFT's ned and surprise time $N(0,\sigma^2)$, curves

mixture theory
characteristic functions

W&A2. WILLIAMS, G. Trevor (ed. by ANCKER, C.J., Jr.), "Stochastic Duels With Displacements (Suppression)," U.S. Army TRADOC Systems Analysis Activity, White Sands Missile Range, New Mexico, TRASANA Technical Memorandum 3-77, March, 1978, 8 pp., DDC No. AO-52146.

FD-CRIFT (both ned)

A miss may be a complete miss or a near miss. A near miss may either be a kill or cause a movement (suppression of fire) which lasts a time which is a CRV (both ned)

Contestant is vulnerable during a displacement (suppression), but cannot return fire

<u>General Solution</u>: P[A]

difference equations
conditional probabilities

21. ZINGER, A., "Concentrated Firing in Many Versus Many Duels," Université du Québec, Montreal, unpublished, July, 1978, 27 pp.

FD - Alternate firing

- Fixed multiple hits to a kill required (different for each side)

<u>General Solutions</u>: $P(A \mid B \text{ starts})$, $P(A \mid A \text{ starts})$, $P(B \mid A \text{ starts})$, $P(B \mid B \text{ starts})$

difference equations